CONTRIBUTIONS TO THE THEORY OF NONLINEAR OSCILLATIONS

VOLUME IV

H. ANTOSIEWICZ
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PRINCETON
PRINCETON UNIVERSITY PRESS
1958

L. C. Card 50-2400

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Printed in the United States of America

PREFACE

The present volume of the Contributions, fourth in the series, covers, like its predecessors, a great variety of topics in non-linear differential equations. The first paper by Kakutani and Markus deals with a differential-difference equation arising in the theory of growth phenomena. Beyond general considerations of the functional properties of the solutions, the authors have obtained very detailed information concerning the oscillations and asymptotic behavior of the solutions. The second paper by Lefschetz, a complement to Barocio's Mexican thesis, contains a rather detailed description of singularities of a pair of analytic differential equations in the plane. The third by Bushaw is a noteworthy contribution in the study of discontinuous forcing terms. The particular point amply covered is the rapidity with which the origin is reached by any solution — an important question in control problems. The paper by de Vogelaere deals with the periodic solutions of Störmer's problem arising in electro-magnetic theory. Slotnick's paper is concerned with the instabilities of Hamiltonian systems. This work continues an investigation initiated by J. Moser. Kyner's contribution relates the theory of periodic surfaces along the line developed by S. Diliberto which was amply described in Contributions III. The paper by Seifert deals with the qualitative behavior of planar differential systems by the method of rotating vector fields. Antosiewicz, in his contribution, gives a survey of the second method of Lyapounov. As is well known, this method has been extensively treated in the Soviet Union but is also acquiring great importance in other areas in view of its elasticity and general power.

The contributions of Mendelson and Bass are concerned with the qualitative behavior of the solutions of non-linear differential systems, with many degrees of freedom, near a critical point. Mendelson investigates the phase portrait near an isolated critical point where one characteristic root is zero and the others have real parts of the same sign. Bass studies the instability near an equilibrium from which repulsive forces act.

A number of these papers have been contributed by various Governmental organizations. These are indicated in connection with each paper.

Solomon Lefschetz

CONTENTS

CONTRIBUTIONS TO THE THEORY OF NONLINEAR OSCILLATIONS

VOL. IV

I. ON THE NON-LINEAR DIFFERENCE-DIFFERENTIAL EQUATION $y'(t) = [A - By(t - \tau)]y(t)$

S. Kakutani and L. Markus

1. INTRODUCTION

If one assumes that the net birth rate $y'(t)/y(t)$ of a population $y(t)$ is the constant coefficient A, then elementary considerations yield an exponential growth (or decay) of population, $y(t) = y(0)e^{At}$. A more feasible mathematical model, for certain discussions, could be obtained by assuming that the birth rate coefficient is diminished by a quantity proportional to the population of the preceding generation. These assumptions lead to the functional equation appearing in the title,

$$y'(t) = [A - By(t - \tau)]y(t), \tag{1}$$

where $\tau > 0$, A and B are real numbers. This delay-differential equation also occurs in the theory of certain servo-mechanisms [3].

If $B = 0$, the resulting differential equation is elementary and we shall henceforth assume $B \neq 0$. We simplify equation (1) by writing $z(t) = B\tau y(\tau t)$. Then

$$z'(t) = [a - z(t - 1)]z(t) \quad , \tag{2}$$

where $a = A\tau$. We shall investigate the functional equation (2) and the results can easily be reinterpreted for (1).

DEFINITION. A solution $z(t)$ of (2) is a real continuous function defined on $0 \leq t < 1 + \epsilon$, $\epsilon > 0$, where $z(t) \in C^{(1)}$ on $1 < t < 1 + \epsilon$, and there satisfies the functional equation (2).

Clearly the constants $z = 0$ and $z = a$ are solutions. It is interesting to note that these are the only solutions of period one since

$z'(t) = [a - z(t)]z(t)$ has no other periodic solutions.

Quite general existence theorems are available [1, 2] for difference-differential equations. However we obtain precise knowledge of the behavior of the solutions of (2) and in our detailed investigations the general theory is not directly applicable.

2. GENERAL PROPERTIES OF THE SOLUTIONS

THEOREM 1. Let $\varphi(t)$, $0 \leq t \leq 1$, be a continuous, real-valued function prescribed as an initial condition. Then there exists a solution $z(t)$ of (2), defined on $0 \leq t < \infty$, for which $z(t) = \varphi(t)$ on $0 \leq t \leq 1$. Moreover $z(t)$ is unique in that each solution of (2), agreeing with $\varphi(t)$ on $0 \leq t \leq 1$, also agrees with $z(t)$ on their common domain of definition.

PROOF. Let $z(t) = \varphi(t)$ on $0 \leq t \leq 1$. Define

$$z(t) = \varphi(1) \exp\left[a(t - 1) - \int_0^{t-1} \varphi(s)ds \right]$$

on $1 \leq t \leq 2$. If $z(t)$ is well-defined on $0 \leq t \leq n$, $n = 2, 3, 4, \ldots$, then define

$$z(t) = z(n) \exp\left[a(t - n) - \int_{n-1}^{t-1} z(s)ds \right]$$

on $n \leq t \leq n + 1$. Clearly $z(t)$ is continuous on $0 \leq t < \infty$ and moreover $z(t) \in C^{(1)}$ on $1 < t < \infty$, except possibly at $t = n$.

At $t = n$,

$$z'(n + 0) = z(n)[a - z(n - 1)]$$

and

$$z'(n - 0) = z(n - 1)[a - z(n - 1)]\exp\left[a - \int_{n-2}^{n-1} z(s)ds \right]$$

$$= z(n)[a - z(n - 1)].$$

Thus $z(t) \in C^{(1)}$ for $1 < t < \infty$ and there satisfies

$$z'(t) = [a - z(t - 1)]z(t).$$

If $w(t)$ is another solution, corresponding to the initial function $\varphi(t)$, then let $t_0 \geq 1$ be the l.u.b. $\{t \mid w(t) = z(t)\}$. But on $t_0 \leq t \leq t_0 + 1$,

$$z(t) = z(t_0) \exp\left[a(t - t_0) - \int_{t_0-1}^{t-1} z(s)ds \right]$$

and

$$w(t) = w(t_0) \exp\left[a(t - t_0) - \int_{t_0-1}^{t-1} w(s)ds \right].$$

Since $w(t_0) = z(t_0)$ and furthermore $w(s) = z(s)$ on $t_0 - 1 \leq s \leq t_0$, we have $w(t) = z(t)$ on $t_0 \leq t \leq t_0 + 1$ which contradicts the existence of the finite number t_0. Thus $w(t) = z(t)$ on their common domain of definition. Q.E.D.

Hereafter, by a solution of (2), we shall mean a solution defined on $0 \leq t < \infty$.

COROLLARY. The solution $z(t) \in C^{(1)}$ on $0 \leq t < \infty$ if and only if $\varphi(t) \in C^{(1)}$ on $0 \leq t \leq 1$ and also $\varphi'(1) = [a - \varphi(0)]\varphi(1)$.

PROOF. If $z(t) \in C^{(1)}$ on $0 \leq t < \infty$ then $\varphi(t) \in C^{(1)}$ on $0 \leq t \leq 1$. Furthermore $z(t)$ satisfies the functional equation (2) at $t = 1 + \epsilon$, $\epsilon > 0$, and thus at $t = 1$. But at $t = 1$, $z(1) = \varphi(1)$, $z'(1) = \varphi'(1)$ and $\varphi'(1) = [a - \varphi(0)]\varphi(1)$.

Conversely if $\varphi(t) \in C^{(1)}$ on $0 \leq t \leq 1$, then $z(t) \in C^{(1)}$ on $0 \leq t < \infty$, except possibly at $t = 1$. But both $z'(1 - 0) = \varphi'(1)$ and $z'(1 + 0) = [a - \varphi(0)]\varphi(1)$ exist and are equal. Thus $z(t) \in C^{(1)}$ on $0 \leq t < \infty$. Q.E.D.

Clearly, if we require $\varphi(t) \in C^{(\infty)}$ on $0 \leq t \leq 1$ and also that the derivatives of $\varphi(t)$ at $t = 1$ be related to those at $t = 0$ by the functional equation (2) and by those equations obtained from (2) by differentiation, then the solution $z(t) \in C^{(\infty)}$ on $0 \leq t < \infty$. It is not apparent whether or not there are analytic solutions of (2). The next theorem shows that there are no (entire) analytic solutions of (2) (other than $z = a$, $z = 0$) which can be expressed in terms of elementary functions.

THEOREM 2. Let $z(t)$ be an entire function of the complex variable t and let $z(t)$ satisfy $z'(t) = [a - z(t-1)]z(t)$. Then, unless $z \equiv 0$ or $z \equiv a$, for each integer k,

$$\max_{|t|=r} |z(t)| = M(r) > \exp \exp \ldots \exp r$$

(k repetitions) for all sufficiently large r.

PROOF. If $z(t_0) = 0$, then $z'(t_0) = 0$ and clearly $z^{(n)}(t_0) = 0$ so that $z \equiv 0$. Thus assume that $z(t)$ has no zeros. If $z(t)$ is of finite order, then by Hadamard's theorem, $z(t) = \exp P(t)$ where $P(t)$ is a polynomial. But then $P'(t) = [a - \exp P(t-1)]$ which is impossible unless $P(t)$ is a constant. But the only constant solutions are $z \equiv 0$ and $z \equiv a$.

Now allow $z(t)$ to be of infinite order. Let $\operatorname{Log} z(t)$ be an entire function such that $\exp \operatorname{Log} z(t) = z(t)$. For any entire function $f(t)$ one knows[1]

$$\max_{|t|=r} |f'(t)| \leq \frac{CR}{(R-r)^2} \{A(R) + f(0)\} \quad ,$$

where

$$A(R) = \max_{|t|=R} \operatorname{Re}\{f(t)\}$$

provided $A(R) \geq 0$, C is a positive constant, and $0 < r < R$. Set $R = 2r$, $f(t) = \operatorname{Log} z(t)$ and we obtain

$$\max_{|t|=r} \left| \frac{d}{dt} \operatorname{Log} z(t) \right| \leq \frac{C_1}{r} \left\{ \max_{|t|=2r} \operatorname{Log} |z(t)| + C_2 \right\} \quad ,$$

where C_1, C_2 are positive constants and the above result applies since

$$\max_{|t|=2r} \operatorname{Log} |z(t)| = \log \left\{ \max_{|t|=2r} |z(t)| \right\} > 0 \quad .$$

Set

$$M(u) = \max_{|t|=u} |z(t)| \quad .$$

---

[1] Dr. J. Wermer suggested this argument.

Then

$$\max_{|t|=r} \left| \frac{z'(t)}{z(t)} \right| \leq \frac{C_3}{r} \log M(2r)$$

for a positive constant C_3. But

$$|z(t)| \leq \left| \frac{z'(t+1)}{z(t+1)} \right| + |a|$$

and so

$$M(r) \leq \frac{C_4}{(r+1)} \log M(2r + 2)$$

for a positive constant C_4.

Now, proceeding by induction, suppose there are no non-constant entire solutions $z(t)$ with $M(r) \leq \exp \exp \ldots \exp r$ (k repetitions), for all large r. If $z(t)$ is an entire solution with

$$M(r) \leq \exp \exp \ldots \exp r$$

($k + 1$ repetitions), then

$$M(r) \leq \frac{C_4}{r+1} \exp \exp \ldots \exp(2r + 2)$$

(k repetitions). Using the same reduction again we have

$$M(r) \leq \frac{C_4}{r+1} \left[\log C_4 - \log(r + 1) + \exp \exp \ldots \exp(4r + 6)\right]$$

($k - 1$ repetitions). But this states that

$$M(r) \leq \exp \exp \ldots \exp r$$

(k repetitions) for all large r. This contradicts the induction hypothesis. Q.E.D.

E. M. Wright [5] has shown that there are real-valued entire solutions for $a > 0$ and, at least for small a, these solutions can be positive on $t \geq 0$.

Returning to the consideration of real solutions, we note that a solution $z(t)$ is nowhere zero for $1 \leq t < \infty$ in case $\varphi(1) \neq 0$.

THEOREM 3. If $\varphi(1) > 0$, then the corresponding solution $z(t) > 0$ for $1 \leq t < \infty$. If $\varphi(1) < 0$, then $z(t) < 0$ for $1 \leq t < \infty$. If $\varphi(1) = 0$, then $z(t) \equiv 0$ for $1 \leq t < \infty$.

PROOF. Consider only the case $\varphi(1) > 0$. Here

$$z(t) = \varphi(1) \exp\left[a(t-1) - \int_0^{t-1} \varphi(s)ds \right] > 0$$

on $1 \leq t \leq 2$. Similarly, if $z(t) > 0$ on $1 \leq t \leq n$, then $z(n) > 0$ and

$$z(t) = z(n) \exp\left[a(t-n) - \int_{n-1}^{t-1} z(s)ds \right] > 0$$

on $n \leq t \leq n+1$. Therefore $z(t) > 0$ on $1 \leq t < \infty$. Similar proofs hold in the other cases $\varphi(1) < 0$ and $\varphi(1) = 0$. Q.E.D.

COROLLARY. Let two solutions $z_1(t)$ and $z_2(t)$ agree on a unit interval $0 \leq t_1 \leq t \leq t_1 + 1$. Then either $z_1(t) = z_2(t)$ on $0 \leq t < \infty$ or else $z_1(t) = z_2(t) = 0$ on $1 \leq t < \infty$.

PROOF. By the argument for uniqueness used in Theorem 1, $z_1(t) = z_2(t)$ on $t_1 \leq t < \infty$. Now $z_i(t)$ $(i = 1$ or $2)$ vanishes at some point on $t_1 \leq t < \infty$ if and only if $z_i(1) = 0$ and then $z_i(t) = 0$ on $1 \leq t < \infty$. Thus either $z_1(t) = z_2(t) = 0$ on $1 \leq t < \infty$ or else neither solution vanishes anywhere on $1 \leq t < \infty$. But then the equation $z(t-1) = a - z'(t)/z(t)$ determines that $z_1(t) = z_2(t)$ on $0 \leq t < \infty$. Q.E.D.

In the following analysis we shall be primarily interested in the case where the intersects of a solution curve $z = z(t)$ with the line $z = a$ form a discrete set.

THEOREM 4. The intersections of the solution curve $z = z(t)$ with the line $z = a$ are discrete on $0 \leq t < \infty$ if and only if there are a finite number of zeros of $\varphi(t) - a$ on $0 \leq t \leq 1$.

PROOF. If the zeros of $z(t) - a$ are discrete on $0 \leq t < \infty$, then, a fortiori, the zeros of $\varphi(t) - a$ are discrete on $0 \leq t \leq 1$.

Conversely, suppose the zeros of $\varphi(t) - a$ are discrete on $0 \leq t \leq 1$. Then, since we may take $\varphi(1) \neq 0$, the zeros of $z'(t)$ on $1 \leq t \leq 2$ are discrete. Thus there are only a finite number of zeros of $z(t) - a$ on $1 \leq t \leq 2$. By an induction argument one shows that there are only a finite number of zeros of $z(t) - a$ on each unit interval $n \leq t \leq n+1$, $n = 1, 2, \ldots$. Q.E.D.

3. THE PRINCIPAL CASE, $a > 0$

THEOREM 5. Let $a > 0$, $\varphi(1) > 0$ and let $z(t)$ be the solution of (2) corresponding to the initial function $\varphi(t)$. Then $0 < m \leq z(t) \leq M < \infty$ on $1 \leq t < \infty$ where

$$M = \max \left\{ \max_{1 \leq t \leq 3} z(t),\ ae^{a} \right\}$$

and

$$m = \min \left\{ \min_{1 \leq t \leq 3} z(t),\ ae^{a-M} \right\} .$$

PROOF. Since $\varphi(1) > 0$, $z(t) > 0$ and $a - z(t) < a$ on $1 \leq t < \infty$. Suppose there exists a $t_1 \geq 3$ with $z(t_1) > M \geq ae^{a}$. Then $z'(t) = [a - z(t - 1)]z(t) < a\, z(t)$ on $t_1 - 1 \leq t \leq t_1$ and $z(t_1 - 1) > a$. Thus $z'(t_1) < 0$ and one sees easily that $z(t)$ is monotone decreasing on $3 \leq t \leq t_1$. But then $z(3) > M$ which contradicts the definition of M. Therefore $z(t) \leq M$ on $1 \leq t < \infty$.

Now $a - z(t) \geq a - M$ on $1 \leq t < \infty$. Suppose $z(t_2) < m \leq ae^{a-M}$ for some $t_2 \geq 3$. Then $z(t_2 - 1) < a$, since $z'(t) \geq (a - M)\, z(t)$ on $t_2 - 1 \leq t \leq t_2$. Thus $z'(t_2) > 0$ and so $z(t)$ is monotonely increasing on $3 \leq t \leq t_2$. Therefore $z(3) < m$ which is a contradiction. Q.E.D.

THEOREM 6. Let $a > 0$ and $\varphi(1) > 0$ and let $z(t)$ be the solution of (2) corresponding to the initial function $\varphi(t)$. Then either

(i) $z(t)$ is asymptotic to $z = a$; that is, $z(t)$ and $z'(t)$ are monotone for large t and

$$\lim_{t=\infty} z(t) = a, \qquad \lim_{t=\infty} z'(t) = 0 ,$$

or

(ii) $z(t)$ oscillates about $z = a$; that is, $z(t) - a$ assumes both positive and negative values for arbitrarily large t.

PROOF. If $z(t) \geq a$ for all large t, then $a - z(t - 1) \leq 0$ and $z'(t) \leq 0$. Thus $z(t)$ decreases monotonely to a limit which is easily seen to be a. Furthermore

$$z''(t) = \{[a - z(t - 1)]^2 - z'(t - 1)\}\, z(t) \geq 0$$

so that $z'(t)$ increases monotonely to a limit which is easily seen to be zero.

If $z(t) \leq a$ for all large t, then $a - z(t - 1) \geq 0$ and $z'(t) \geq 0$. Thus $z(t)$ increases to the limit a. Also $z'(t) \geq [a - z(t)]\, z(t) \geq [a - z(t)]^2$ for large t so that $z''(t) \leq 0$. Thus $z'(t)$ decreases monotonely to the limit zero.

If neither $z(t) \geq a$ nor $z(t) \leq a$ for all large t, then $z(t)$ oscillates about the line $z = a$. Q.E.D.

As we shall later see, each of these alternatives is possible for certain values of the parameter a. However, before determining which values of the parameter a produce which behaviors, we shall investigate the qualitative form of the oscillatory solutions.

THEOREM 7. Let $a > 0$, $\varphi(1) > 0$ and $z(t)$, the solution of (2) corresponding to $\varphi(t)$, oscillate about $z = a$. Assume the zeros of $z(t) - a$ are a discrete set on $0 \leq t < \infty$. Then, for sufficiently large t, each zero of $z(t) - a$ is simple and there is exactly one zero of $z'(t)$ between consecutive zeros of $z(t) - a$.

PROOF. Let the number of zeros of $z(t) - a$ on $n \leq t \leq n + 1$, $n = 1, 2, 3, \ldots$, be $k(n)$. Then define the number $\bar{k}(n)$ of potential zeros to be $\bar{k}(n) = k(n)$ if the last zero $t_\ell = n + 1$ or if

$$\frac{d}{dt}\,|z(t) - a| > 0$$

on $\max(n, t_\ell) < t < n + 1$ and define $\bar{k}(n) = k(n) + 1$ in all other cases. We show that $\bar{k}(n)$ is a non-increasing function of n.

Suppose $\bar{k}(n) = k(n)$ and $t_\ell = n + 1$. Then there are at most $k(n) - 1$ bend points of $z(t) - a$ in $n + 1 \leq t < n + 2$. Thus $k(n + 1) \leq k(n)$ and for equality one must have a situation in which $\bar{k}(n + 1) = k(n + 1)$. Therefore in this case $\bar{k}(n + 1) \leq \bar{k}(n)$.

Next suppose $\bar{k}(n) = k(n)$ and

$$\frac{d}{dt}\,|z(t) - a| > 0$$

on $\max(n, t_\ell) < t < n + 1$. Then on $n + 1 \leq t \leq n + 2$ there are at most $k(n)$ bend points, at least one of which must occur before the first zero of $z(t) - a$. Thus $k(n + 1) \leq k(n)$. If $k(n + 1) = k(n)$, then no zero of $z'(t)$ occurs following the last zero of $z(t) - a$ on $n + 1 \leq t \leq n + 2$,

and hence $\bar{k}(n+1) = k(n+1)$. Therefore $\bar{k}(n+1) \leq \bar{k}(n)$ in this case.

Finally suppose $\bar{k}(n) = k(n) + 1$. There are at most $k(n)$ bend points of $z(t) - a$ on $n + 1 \leq t \leq n + 2$ and thus $k(n+1) \leq k(n) + 1$. But if $k(n+1) = k(n) + 1$ the situation is such that $\bar{k}(n+1) = k(n+1)$. Therefore $\bar{k}(n+1) \leq \bar{k}(n)$ in all cases.

Now let

$$\lim_{n=\infty} \bar{k}(n) = \bar{k}$$

and say, for $n > N$, $\bar{k}(n) = \bar{k}$. Suppose on $N + r \leq t \leq N + r + 1$, $r = 1, 2, 3, \ldots$, there are at least two zeros of $z'(t)$ on the open interval between two successive zeros of $z(t) - a$. Then there are at least $\bar{k}(N + r)$ zeros of $z'(t)$ on $N + r \leq t \leq N + r + 1$. But then there are at least $\bar{k}(N + r)$ zeros of $z(t) - a$ on $N + r - 1 \leq t \leq N + r$, that is, $k(N + r - 1) \geq \bar{k}(N + r)$. Furthermore equality holds only when $\bar{k}(N + r - 1) = k(N + r - 1) + 1$. Thus $\bar{k}(N + r - 1) > \bar{k}(N + r)$ which contradicts the property that $\bar{k}(n) = \bar{k}$ for $n \geq N$.

A similar argument shows that a double root of $z(t) - a$ also results in a decrease in $\bar{k}(n)$ and so can not occur for $n \geq N$. Q E.D.

COROLLARY. Let $a > 0$, $\varphi(1) > 0$ and let the solution $z(t)$ oscillate about $z = a$. Let $\varphi(t) - a \in C^{(1)}$ on $0 \leq t \leq 1$ have at most one potential zero (in particular if $\varphi(t) - a \neq 0$), then the interval from each zero of $z(t) - a$, $t \geq 1$, to the following extremum is of length one. The interval from an extremum of $z(t) - a$, of amplitude $\epsilon > 0$, to the following zero is of length

$$d_\epsilon > \frac{1}{\epsilon} \log (1 + \epsilon/a) \quad .$$

If $a > 1$ and if ϵ is sufficiently small, then

$$\frac{1}{a}\left[1 - \epsilon/2a\right] < d_\epsilon < 2 \quad .$$

PROOF. If $\varphi(t) - a$ has no zeros on $0 \leq t \leq 1$, then $z'(t) \neq 0$ on $1 \leq t \leq 2$ and $\bar{k}(1) = 1$. If $\varphi(t) - a$ has one zero followed by a zero of $\varphi'(t)$ on $0 \leq t \leq 1$, then again $z'(t)$ has one zero on $1 \leq t \leq 2$ and $\bar{k}(1) = 1$.

Now consider an extremum where $z'(t_o) = 0$ on $n \leq t_o \leq n + 1$, $n = 1, 2, \ldots$. Then on $n - 1 \leq t \leq n$ there is one, and thus only one, zero of $z(t) - a$, since $\bar{k}(n-1) = k(n-1) = 1$. If there were another extremum on $t_o - 1 \leq t \leq t_o$, then $\bar{k}(n) \geq 2$ which is impossible. Therefore

$z(t) - a$ is strictly monotone on $t_0 - 1 \leq t \leq t_0$.

Let t_1 be an extremum where $z(t_1) = a + \epsilon$. Then

$$z(t) > (a + \epsilon)e^{-\epsilon(t-t_1)}$$

for $t_1 < t < t_1 + d_\epsilon$. But for

$$t - t_1 = \frac{1}{\epsilon} \log (1 + \epsilon/a) ,$$

$$(a + \epsilon)e^{-\epsilon(t-t_1)} = (a + \epsilon) \left(\frac{a}{a+\epsilon}\right) = a \quad .$$

Thus $d_\epsilon > \frac{1}{\epsilon} \log (1 + \epsilon/a)$.

For a minimum where $z(t_2) = a - \epsilon$,

$$z(t) < (a - \epsilon)e^{\epsilon(t-t_2)} < a$$

when

$$(t - t_2) \leq \frac{1}{\epsilon} \log \left(\frac{a}{a-\epsilon}\right) \quad .$$

Thus

$$d_\epsilon > \min \left\{\frac{1}{\epsilon} \log (1 + \epsilon/a), \quad \frac{1}{\epsilon} \log \frac{a}{a-\epsilon}\right\} = \frac{1}{\epsilon} \log (1 + \epsilon/a) \quad .$$

If ϵ is small, say $\epsilon < a$, then

$$d_\epsilon > \frac{1}{\epsilon} \left[\epsilon/a - \frac{1}{2} (\epsilon/a)^2 + \frac{1}{3} (\epsilon/a)^3 - \dots\right] > \frac{1}{\epsilon} \left[\epsilon/a - \frac{1}{2} (\epsilon/a)^2\right] \quad .$$

Thus

$$d_\epsilon > \frac{1}{a} \left[1 - \frac{1}{2} \frac{\epsilon}{a}\right] \quad .$$

Now take $a > 1$. Suppose that following a maximum $z(t_1) = a + \epsilon$, $z(t) \geq a$ on $t_1 \leq t \leq t_1 + 2$. But then on $t_1 + 1 \leq t \leq t_1 + 2$, $- z'(t) > \epsilon_1 z(t)$ and

$$z(t) < (a + \epsilon_1)e^{-\epsilon_1(t-t_1-1)}$$

where $\epsilon_1 = z(t_1 + 1) - a$. Then $z(t_1 + 2) < (a + \epsilon_1)(1 - \epsilon_1 + \epsilon_1^2/_2) < a$ which is a contradiction. In this case $d_\epsilon < 2$.

Similarly, after a minimum where $z(t_2) = a - \epsilon$, suppose $z(t) \leq a$ on $t_2 \leq t \leq t_2 + 2$. Then on $t_2 + 1 \leq t \leq t_2 + 2$, $z'(t) > \epsilon_2 z(t)$ and

$$z(t) > (a + \epsilon_2)e^{\epsilon_2(t-t_2-1)}$$

where $\epsilon_2 = a - z(t_2 + 1)$. Then $z(t_2 + 2) > (a - \epsilon_2)(1 + \epsilon_2^{\sim}) > a$. Thus here also $d_\epsilon < 2$. Q.E.D.

The most interesting oscillations are those which are strictly monotone from each zero of $z(t) - a$, for a unit length, until the following extremum. Such an oscillation is concave up on $t_1 + 1 < t < t_2 + 1$, where t_1 and t_2 are successive maxima and minima respectively. Thus on

$$t_2 - \frac{1}{\epsilon} \log (1 + \epsilon/a) \leq t \leq t_2 + 1$$

or on

$$t_2 - \frac{1}{2a} \leq t \leq t_2 + 1 ,$$

where $\epsilon < a$ is the amplitude of the maximum, the solution $z(t)$ is concave upwards.

THEOREM 8. Let $0 < a \leq 1$ and let $z(t) - a$ oscillate with discrete zeros. Then the oscillations are damped and

$$\lim_{t=\infty} z(t) = a \quad .$$

PROOF. Let $\epsilon_1, \epsilon_2, \ldots$ and $\delta_1, \delta_2, \ldots$ be the amplitudes of successive maxima and minima respectively, with δ_1 occurring before ϵ_1 and we consider only large t so that the oscillations have the form described in Theorem 7.

Suppose ϵ_{1+k} is the maximum which follows by one unit the first zero of $z(t) - a$ after δ_1. Then clearly ϵ_{n+k} is the maximum following by one unit the first zero of $z(t) - a$ after δ_n, $n = 1, 2, \ldots$. Also δ_{n+k+1} is the minimum following by one unit the first zero of $z(t) - a$ after ϵ_n.

A trivial estimate yields

$$a + \epsilon_{n+k} < ae^{\delta_n}$$

and

$$a - \delta_{n+k+1} > ae^{-\epsilon_n} \quad .$$

But then

$$\epsilon_n < a \left\{\exp a \left[1 - e^{-\epsilon_{n-2k-1}}\right] - 1\right\} \quad .$$

We show that $\epsilon_n < \epsilon_{n-2k-1}$.

Consider the function

$$f(\epsilon) = a\left\{\exp a\left[1 - e^{-\epsilon}\right] - 1\right\} .$$

Now $f(0) = 0$ and

$$f'(\epsilon) = a^2 \exp a\left[1 - e^{-\epsilon} - \epsilon/a\right] .$$

Thus, for $\epsilon > 0$,

$$f'(\epsilon) < a^2 \exp a\left[1 - \epsilon - e^{-\epsilon}\right] < a^2 \leq 1 .$$

Thus $\epsilon \longrightarrow f(\epsilon)$ is a strict contraction of the half-line $\epsilon > 0$ onto itself. Since $\epsilon_n < f(\epsilon_{n-2k-1}) < \epsilon_{n-2k-1}$ we have

$$\lim_{n=\infty} \epsilon_{n_1+(2k+1)n} = 0$$

where $n_1 = 1, 2, \ldots, 2k$. But then

$$\lim_{n=\infty} \epsilon_n = 0 .$$

Since

$$\delta_{n+k+1} < a\left[1 - e^{-\epsilon_n}\right], \quad \lim_{n=\infty} \delta_n = 0 .$$

Therefore

$$\lim_{t=\infty} z(t) = a .$$

Q.E.D.

Thus for $0 < a \leq 1$, $\varphi(1) > 0$, every solution $z(t)$ is either asymptotic to $z = a$ or is a damped oscillation and thus

$$\lim_{t=\infty} z(t) = a .$$

It seems likely that for $a > 1$, some of the oscillations are not damped (there are only oscillations, see Theorem 9) and do not approach a limit value.

THEOREM 9. Let $a > 1/e$ and $\varphi(1) > 0$. Then no solution $z(t)$ is asymptotic to $z = a$ (except $z(t) \equiv a$).

PROOF. If $z'(t) = [a - z(t-1)]z(t)$, let $y(t) = \frac{1}{a} z(t)$ and then $y(t)$ is a solution of $y'(t) = a[1 - y(t-1)]y(t)$ which is asymptotic to $y = 1$ if and only if $z(t)$ is asymptotic to $z = a$. Let $\zeta(t) = 1 - y(t)$ and then $\zeta(t) \searrow 0$ with $\zeta'(t) \nearrow 0$ in case $z(t) \nearrow a$ or $y(t) \nearrow 1$ as $t \longrightarrow \infty$.

Suppose $z(t)$ is asymptotic to $z = a$ from below. Now

$$\frac{-\zeta'(t)}{\zeta(t)} = a\,\frac{\zeta(t-1)}{\zeta(t)}\,[1 - \zeta(t)]$$

and we take t so large that $a[1 - \zeta(t)] = ay(t) > a' = a - \epsilon > \frac{1}{e}$, for a small $\epsilon > 0$. Write

$$R(t) = \frac{-\zeta'(t)}{\zeta(t)} > 0$$

and then

$$R(t) > a' \exp \int_{t-1}^{t} R(s)ds \quad .$$

Say on $n - 1 \leq s \leq n$, $R(s) > M > 0$. Then on $n \leq s \leq n + 1$, $R(n) > a'e^M > M$ and (since there is no point at which $R(s) = a'e^M$) $R(s) > a'e^M$. By induction on $n + k \leq s \leq n + k + 1$ we have $R(s) > a'\exp R(n + k - 1)$.

Consider the number sequence $C_1 = a'e^M$, $C_2 = a' \exp a'e^M$, ..., $C_{n+1} = a' \exp C_n$, ... By observing that the graph of $a'e^x$ lies above that of x, we note that $C_{n+1} > C_n$ and

$$\lim_{n=\infty} C_n = +\infty \quad .$$

Thus whenever

$$a > 1/e, \quad \frac{-\zeta'(t)}{\zeta(t)} \longrightarrow +\infty$$

as $t \longrightarrow \infty$.

Let T be so large that

$$\frac{-\zeta'(t)}{\zeta(t)} > k$$

whenever $t > T - 1$ where k is a large constant specified later. Since on $T \leq t \leq T + 1$, $R(t) > k$, then

$$R\left(T + \frac{1}{2} + \tau_1\right) > a'e^{k/2} = \alpha_1$$

where $0 \leq \tau_1 \leq \frac{1}{2}$. Also

$$R\left(T + \frac{3}{4} + \tau_2\right) > a' \exp\left(\frac{k}{2} + \frac{1}{4}\,a'e^{k/2}\right) = \alpha_2$$

where $0 \leq \tau_2 \leq \frac{1}{4}$. Similarly

$$R\left(T + \frac{2^n-1}{2^n} + \tau_n\right) > a' \exp\left(\frac{k}{2} + \frac{1}{2^2} a' e^{k/2} + \dots + \frac{1}{2^n} \alpha_{n-1}\right) = \alpha_n \quad ,$$

where $0 \leq \tau_n \leq \frac{1}{2^n}$.

Consider the number sequence

$$\alpha_1 = a' e^{k/2}, \ \alpha_2 = \alpha_1 e^{\alpha_1/4}, \ \dots, \ \alpha_{n+1} = \alpha_n e^{\alpha_n/2^{n+1}} \ \dots \ .$$

Take $k = 2 \log 100 + 2$ so that $\alpha_1 > 100$. Then one easily calculates $\alpha_n > 100^n$ so that

$$\lim_{n=\infty} \alpha_n = +\infty \quad .$$

But this implies that $R(t)$ is not bounded on $T \leq t \leq T + 1$. However this contradicts the assumption that $\zeta(t) \neq 0$ and thus the asymptotic approach from below is impossible.

Next we show that there are no asymptotic solutions to $y = 1$ from above when $a > 1/e$. Here let $\xi(t) = y(t) - 1$ and suppose $\xi(t) \searrow 0$, $\xi'(t) \nearrow 0$, and

$$\frac{-\xi'(t)}{\xi(t)} = a \ [1 + \xi(t)] \ \frac{\xi(t-1)}{\xi(t)} \quad .$$

Also take t so large that $a[1 + \xi(t)] > a > 1/e$ and let

$$P(t) = \frac{-\xi'(t)}{\xi(t)} > 0 \quad .$$

Again

$$P(t) = a \exp \int_{t-1}^{t} P(s)ds$$

and $P(t) \longrightarrow +\infty$ as $t \longrightarrow \infty$. Take $P(t) > k$ whenever $t > T - 1$ and as above $P(t)$ is not bounded on $T \leq t \leq T + 1$ which contradicts the assumption of an asymptotic solution with $\xi(t) \neq 0$. Q.E.D.

Theorems 8 and 9 show that if $1/e < a \leq 1$ and $\varphi(1) > 0$, then every solution is a damped oscillation tending to the limit value of a.

The next result is the complement of Theorem 9 and proves the existence of asymptotic solutions for $0 < a \leq 1/e$.

THEOREM 10. Let $0 < a \leq 1/e$ and $\varphi(1) > 0$. Let the interval between the zeros of $z(t) - a$ be at least one, for large t. Then the solution $z(t)$ is asymptotic to $z = a$.

PROOF. Suppose $z(t)$ has an oscillation, necessarily damped, about $z = a$. Then $y(t) = \frac{1}{a} z(t)$ is a solution of $y'(t) = a[1 - y(t-1)]y(t)$, and has a damped oscillation about $y = 1$. Moreover the interval from each zero of $y(t) - 1$ (for large $t > T$) to the following extremum is of length one.

Consider the family of curves $w = 1 - Be^{-t}$ for $B > 0$. We shall show that there exists a curve $\bar{w}(t)$ of this family which is tangent to $y(t)$ between the minimum at $t_0 > T$ and the following zero t' of $y(t) - 1$.

Along $y(t)$, define $B(t)$, $t_0 \leq t < t'$, as the parameter value of the curve of the family through $(t, y(t))$. Then near t_0, $B(t)$ is monotone increasing. Also

$$\lim_{t=t'} B(t) = 0 \quad .$$

Define

$$\bar{B} = \max_{t_0 \leq t < t'} B(t)$$

and let $\bar{w}(t) = 1 - \bar{B}e^{-t}$. Let t_1 be the abscissa of the first intersection of $y(t)$ and $\bar{w}(t)$ on $t_0 \leq t < t'$. Then $\bar{w}(t_1) = y(t_1)$ and also $\bar{w}'(t_1) = y'(t_1)$. Also $\bar{w}(t) < y(t)$ on $t_1 - 1 \leq t < t_1$. But then

$$\frac{\bar{B}e^{-t_1}}{1 - \bar{B}e^{-t_1}} = \frac{\bar{w}'(t_1)}{\bar{w}(t_1)} = \frac{y'(t_1)}{y(t_1)} = a[1 - y(t_1 - 1)] < a[1 - \bar{w}(t_1 - 1)] \quad .$$

Thus

$$\frac{\bar{B}e^{-t_1}}{1 - \bar{B}e^{-t_1}} < a\bar{B}e^{-t_1}e$$

and

$$a > \frac{1}{e} \frac{1}{1 - \bar{B}e^{-t_1}} > 1/e \quad .$$

But this contradicts the hypothesis $0 < a \leq 1/e$ and therefore there are no oscillatory solutions with half-wave length greater than one. Q.E.D.

In particular if $0 < a \leq 1/e$, $\varphi(1) > 0$, and $\varphi(t) \neq a$ on $0 \leq t \leq 1$, then $z(t)$ is necessarily asymptotic to $z = a$.

Finally we show that for $a > 0$ only positive solutions are bounded.

THEOREM 11. Let $a > 0$ and $\varphi(1) < 0$. Then

$$\lim_{t=\infty} z(t) = -\infty \quad .$$

In fact, $e^{\alpha t} = o|z(t)|$ as $t \longrightarrow \infty$ for each $\alpha > 0$.

PROOF. Since $\varphi(1) < 0$, then $z(t) < 0$ for $t \geq 1$. Moreover $z'(t) = [a - z(t-1)]z(t) < 0$ and $z(t)$ is monotonely decreasing for $t \geq 1$. Since $[a - z(t-1)] > a$,

$$\lim_{t=\infty} z(t) = - \infty .$$

Given $\alpha > 0$, there is a T such that, for $t > T$, $[a - z(t-1)] > \alpha$ and thus $e^{\alpha t} = o|z(t)|$ as $t \longrightarrow \infty$. Q.E.D.

4. THE REMAINING CASES, $a \leq 0$

The next theorem shows that the only non-trivial behavior arises when both a and $z(t)$ are negative.

THEOREM 12. If $a \leq 0$ and $\varphi(1) > 0$, then

$$\lim_{t=\infty} z(t) = 0 \quad .$$

If $a = 0$ and $\varphi(1) < 0$ then

$$\lim_{t=\infty} z(t) = -\infty$$

and, in fact, $e^{\alpha t} = o|z(t)|$ as $t \longrightarrow \infty$ for each $\alpha > 0$.

PROOF. If $a \leq 0$ and $\varphi(1) > 0$, then $z(t) > 0$ and $[a - z(t-1)] < a \leq 0$ for $t \geq 2$. Therefore $z(t)$ decreases monotonely and clearly

$$\lim_{t=\infty} z(t) = 0 \quad .$$

If $a = 0$ and $\varphi(1) < 0$, then $z(t) < 0$ and

$z'(t) = -z(t-1)z(t)$ so $z(t)$ decreases monotonely for $t \geq 2$. Also

$$\lim_{t=\infty} z(t) = -\infty \quad .$$

Then given $\alpha > 0$, choose T such that, for $t > T$, $-z(t-1) > \alpha$. Therefore $e^{\alpha t} = o|z(t)|$ as $t \longrightarrow \infty$. Q.E.D.

THEOREM 13. Let $a < 0$ and $\varphi(1) < 0$ and assume $z(t) \not\equiv a$. Then

1) if $z(t) \leq a$ for a unit interval,

$$\lim_{t=\infty} z(t) = -\infty \ ;$$

2) if $z(t) \geq a$ for a unit interval,

$$\lim_{t=\infty} z(t) = 0 \ ;$$

3) in all other cases $z(t) - a$ oscillates and is both positive and negative in each unit interval.

PROOF. By the corollary to Theorem 3, $z(t) \not\equiv a$ on any unit interval. In cases 1) and 2) the conclusions follow just as in Theorem 12. The three cases clearly exhaust the possible behaviors for $z(t)$. Q.E.D.

It is not known whether the oscillations of case 3) above can occur. However if, on $0 \leq t \leq 1$, $\varphi(t) \neq a$, or has just one potential zero, then case 3) is impossible (cf. Theorem 7).

5. ADDENDUM

After this paper was submitted, the authors noted the closely related work of E. M. Wright [4, 5, 6] on the same functional equation. There is some overlap in the results but in certain interesting aspects the papers complement one another. For example, Wright solves the problem treated in our Theorem 2 by finding analytic solutions on the whole real line, cf. [5 and 6, thm 9]. Also, he strengthens our Theorem 8, cf. [6, thm 3, 4], by replacing our estimate of 1 by $3/2$ and he suggests $\pi/2$ as best possible. On the other hand, our principal Theorems 9 and 10 on asymptotic solutions seem new.

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II. ON THE CRITICAL POINTS OF A CLASS OF DIFFERENTIAL EQUATIONS

Solomon Lefschetz

1. This Note is closely related to Barocio's Mexican thesis, and we shall largely use its terminology and notations with slight variants. In particular

$[x]_p$, $[x, y]_p$ denote convergent power series in x or x and y beginning with terms of degree $\geqq p$;

$E[x]$, $E[x, y]$ are convergent power series such that $E(0) = E(0, 0) = 1$;

TO-curve denotes a path leading to or away from the origin in a definite direction.

The systems which we study are of the general type:

$$\left\{\begin{aligned} \frac{dx}{dt} &= -(y - C(x)) \\ \frac{dy}{dt} &= \left[y^2 - 2A(x)y + B(x)\right]E(x, y) \end{aligned}\right. \tag{1.1}$$

$$A = [x]_1; \quad B, C = [x]_2 \quad .$$

All systems with both characteristic roots zero but with terms of the first degree not all zero are reducible to this form. Furthermore systems (1.1) are characterized as those whose paths are orthogonal to the paths of a Bendixson system (one characteristic root $\neq 0$), a result due to Barocio. This orthogonality already enables one to obtain considerable information regarding the local phase-portrait of (1.1) from the known local phase-portrait of Bendixson systems. However while the latter case only possesses hyperbolic sectors, (1.1) may well have a set of nested ovals. Such a system was in fact explicitly obtained by Courtney Coleman (1955 Princeton thesis). How many such sets may arise has as yet been an open question. The following proposition which we shall now prove answers the question:-

(1.2) THEOREM. A system with both characteristic roots zero but with first degree terms not all zero possesses at most a single sector of nested ovals (= s. n. o.). In the reduced form (1.1) this single sector if it exists must be crossed by the y axis.

The proof will consist primarily in showing that in the form (1.1) no s. n. o. can be on just one side of the y axis. Once this is proved the full theorem follows with ease.

2. In our treatment we shall pass to the system equivalent to (1.1)

$$\frac{dy}{dx} = \frac{[y^2-2A(x)y+B(x)]E(x,y)}{-(y-C(x))} \quad . \tag{2.1}$$

We shall first discuss the possible existence of an s. n. o. to the right of the y axis. In that region there may exist branches issued from the origin where $\frac{dy}{dt} = 0$. If there are such branches there are two of them and we denote them by Γ_H^1, Γ_H^2. In our region there exists always a branch Γ_V where $\frac{dx}{dt} = 0$.

The branches Γ_H are jointly given by

$$y^2 - 2A(x)y + B(x) = 0 \quad . \tag{2.2}$$

If $\Delta = A^2 - B = [x]_2$, then the two branches are given by

$$y = A(x) \pm \sqrt{\Delta} \quad . \tag{2.3}$$

Now upon drawing the various sketches corresponding to the Γ_H^1, Γ_V branches to the right of the y axis, we readily find that the only disposition that might give rise to an s. n. o. to the right of Oy is the one of Figure 1: the Γ branches in the first quadrant and the Γ_H^1 above Γ_V.

This figure is drawn under the following conventions adopted by Barocio: the Γ_H branches are in dotted lines and Γ_V is a continuous line.

Now Figure 1 is only compatible with $\Delta = \alpha^2x^{2k}E(x)$, or else $\Delta(x) = \alpha^2x^{2k+1}E(x)$. In order that the two Γ_H branches be in the first quadrant we must have one of the following two systems of representations for our branches:

*

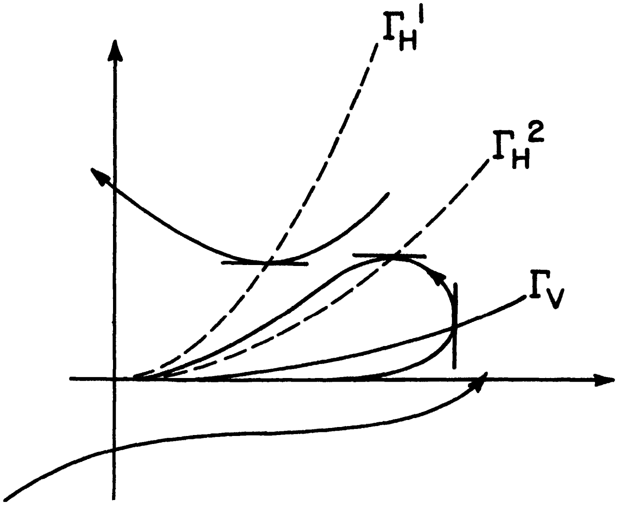

FIGURE 1

I.
$$\Gamma_H^1 : y = ax^pE_1(x) \quad ,$$
$$\Gamma_H^2 : y = bx^qE_2(x) \quad ,$$
$$\Gamma_V : y = cx^rE_3(x) \quad ,$$
$$p \leqq q \leqq r; \quad a, b > 0, \quad c \geqq 0.$$

Observe that $c = 0$ means that Γ_V is the x axis.

II.
$$\Gamma_H^1 : y = (x) + ax^{q + 1/2}E_1\left(x^{1/2}\right)$$
$$\Gamma_H^2 : y = (x) - ax^{q+1/2}E_2\left(x^{1/2}\right)$$
$$\varphi = \alpha x^p + \dots + \rho x^q, \ a > b, \ \alpha > 0;$$
$$\Gamma_H^3 : y = cx^rE_3(x);$$
$$r \geqq p, \quad \text{or else} \quad r = p \quad \text{and} \quad c \leqq \alpha \quad .$$

The types I and II will have to be examined separately.

In the sequel it will be simpler to omit units here and there, at the cost of replacing $=$ by $\doteq$. Thus we will write for example for Γ_H^1 under I:
$$\Gamma_H^1 : y \doteq ax^p \quad .$$

By the <u>order of a TO-curve</u> or simply <u>order</u> we shall mean the order of $y(x)$ on the curve.

Our general method will consist in first finding the possible orders of TO-curves by means of the Newton polygon. Then if μ is such an order we apply the transformation $y = x^{\mu} y_1$. It will turn out that μ is always an integer. The transformation replaces the given equation by a system

$$\frac{dy_1}{dx} = \frac{A(x,y_1)}{x^r B(x,y_1)}$$

with $x = 0$ as a solution. The images of the TO-curves of order μ can only be solutions tending to critical points $P, Q, \ldots,$ other than the origin on the y_1 axis. These are given by the critical equation

$$A(0, y_1) = 0 \quad ,$$

and among them those corresponding to ends of an s. n. o. must be nodes. The strict saddle points are thus to be eliminated at the outset.

3. Consider first case I. The equation (2.1) becomes

$$\text{(3.1)} \qquad \frac{dy}{dx} \doteq \frac{(y-ax^p)(y-bx^q)}{-(y-cx^r)}, \quad p \leqq q \leqq r \quad .$$

The possible orders are found to be $\mu = p + 1, q$. The s. n. o. require orders between p and q and beyond r. We consider the various possibilities.

Ia. $p = q$. This is only compatible with $r = p, p + 1$. Here $\mu = p, p + 1$. Testing first $\mu = p$ (a necessary order) we find

$$\text{(3.2)} \qquad \frac{dy_1}{dx} \doteq \frac{x(y_1-a)(y_1-b) + py_1(y_1-cx^{r-p})}{-x(y_1-cx^{r-p})} \quad .$$

If $r = p + 1$ the only critical value is $y_1 = 0$ and so this is out. If $r = p$ we have $y_1 = c$ as critical value $\neq 0$. The transformation $y_1 - c = y^*$ replaces (3.2) by

$$\text{(3.3)} \qquad \frac{dy^*}{dx} = \frac{pcy^* + \ldots}{-x(y^*+dx+\ldots)} \quad .$$

Hence $P : y_1 = c$ is a Bendixson point. As it is the only critical point ends of s. n. o. must be imaged in arcs from and to P. This would make

of P a point with an s. n. o. and this is ruled out for a Bendixson point. Hence there is no s. n. o. in the present case.

When $q > p$ we have several possibilities.

Ib. $q \geqq p + 2$. This requires $\mu = p + 1, q$, and hence $r = q$. Taking first $\mu = p + 1$, $y = x^{p+1}y_1$ we find

$$\frac{dy_1}{dx} \doteq \frac{(xy_1-a)(y_1-bx^{q-p-1}) + (p+1)y_1(y_1-cx^{q-p-1})}{-x(y_1-cx^{q-p-1})} \quad . \tag{3.4}$$

The critical equation is

$$(p + 1)y_1^2 - ay_1 = 0 \quad . \tag{3.5}$$

Its only root $\neq 0$ is $y_1 = \frac{a}{p+1} = \alpha$. Setting $y_1 - a = y^*$ there follows

$$\frac{dy^*}{dx} = \frac{\lambda x + ay^* + \cdots}{-\alpha x + \cdots} \quad . \tag{3.6}$$

Hence this point is a saddle-point and so it is ruled out.

For $\mu = q$ and $y = x^q y_1$ we have

$$\frac{dy_1}{dx} \doteq \frac{(y_1x^{q-p}-a)(y_1-b) + qy_1x^{q-p-1}(y_1-c)}{-x^{q-p}(y_1-c)} \quad . \tag{3.7}$$

The only critical value is $y_1 = b$. Setting $y_1 - b = y^*$ there follows

$$\frac{dy^*}{dx} = \frac{-ay^* + \cdots}{-x^{q-p}(y^*+b-c)} \quad , \tag{3.8}$$

a Bendixson critical point and as before this rules out an s. n. o.

Ic. $q = p + 1$. Hence $r = p + 1$ and we only have $\mu = p + 1$, hence $y = x^{p+1}y_1$. This time

$$\frac{dy_1}{dx} \doteq \frac{(y_1x-a)(y_1-b) + (p+1)y_1(y_1-c)}{-x(y_1-c)} \quad . \tag{3.9}$$

The critical point equation is

$$f(y_1) = (p + 1)y_1(y_1 - c) - a(y_1 - b) = 0 \quad .$$

Suppose first $b \neq c$. Since Γ_H^2 is above Γ_V necessarily $b > c$. Since some TO-curves must be above Γ_H^2 and some below Γ_V, b must be between the roots of $f(y_1)$ if there is to be an s. n. o. This requires that $f(b) < 0$. However $f(b) = (p + 1)b(b - c) > 0$ and so there is no s. n. o. here.

Suppose now $b = c$. More generally we may have representations

$$\Gamma_H^1 : y = \alpha(x), \quad \Gamma_H^2 : y = \beta(x), \quad \Gamma_V : y = \gamma(x) ,$$

where β, γ are of order $p + 1$, $\beta - \gamma = \delta$ is of order $s > p + 1$, and if $\delta = dx^s + \dots$, then $d > 0$. Thus here

$$\frac{dy}{dx} \doteq \frac{(y-\alpha)(y-\beta)}{-(y-\gamma)} .$$

The regular transformation

$$x = x, \quad y = z + \beta(x)$$

replaces topologically the right of the y axis and its s. n. o.'s by the right of the z axis and its s. n. o.'s. Thus it will be sufficient to eliminate the latter. Now

$$\begin{aligned} \frac{dz}{dx} &\doteq \frac{(z-(\alpha-\beta))z + \beta' \cdot (z+\delta)}{-(z+\delta)} \\ &\doteq \frac{z^2 - \epsilon(x)z + \beta'\delta}{-(z+\delta)} \qquad (3.10) \\ &\doteq \frac{(z-\zeta_1(x))(z-\zeta_2(x))}{-(z+\delta)} . \end{aligned}$$

Notice that since

$$\beta'\delta = (p + 1)bdx^{p+q} + \dots ,$$

$\beta'\delta > 0$ for x small. Hence if s_1, s_2 are the orders of ζ_1, ζ_2 then $s_1 + s_2 = p + s$ and ζ_1, ζ_2 have the same signs for x small. Since δ is then also positive, (3.10) is like our initial system but with Γ_V below Ox. If ζ_1, ζ_2 are > 0, Γ_H^1 and Γ_H^2 are above Ox and an elementary sketch excludes at once an s. n. o.

Suppose now ζ_1, ζ_2 both < 0. The change of variable $z = -z_1$ replaces (3.10) by

$$\frac{dz_1}{dx} = \frac{(z_1+\zeta_1)(z_1+\zeta_2)}{-(z_1-\zeta)}$$

where now $-\zeta_1$, $-\zeta_2$, $\delta > 0$ for x small and their orders are still s_1, s_2, s with $s_1 + s_2 = p + s$. Suppose $s_1 \leqq s_2$. The only possibility for an s. n. o. is $s_2 = s_1 + 1 = s$. This yields $s = p + 1$ which contradicts $s > p + 1$ and eliminates the s. n. o. in the present instance.

To sum up then no s. n. o. may arise under case I.

4. Passing now to case II of No. 3, a first change of variables: $x = u^2$ will yield

$$\frac{dy}{du} \doteq \frac{2u(y-\Psi(u)-au^{2q+1})(y-\Psi(u)+au^{2q+1})}{-(y-cu^{2r})}$$

$$\Psi = \alpha u^{2p} + \dots + \beta u^{2q}; \quad q \geqq p; \quad \alpha,\ a > 0 \quad .$$

From Figure 1 we know that an s. n. o. requires TO-curves between the two Γ_H branches and so of order $2p$, and also that $c > 0$. We apply then the transformation

$$u = u, \quad y = u^{2p}y_1$$

and obtain

$$\frac{dy_1}{du} \doteq 2\,\frac{u^2(y_1-\alpha)^2 + py_1(y_1-cu^{2(r-p)})}{-u(y_1-cu^{2(r-p)})} \quad .$$

There are now two possibilities:

(a) $r > p$. The only critical point is at the origin. This means that possible TO-curves are of order $> p$ and it excludes an s. n. o.

(b) $r = p$ and hence $c \neq 0$. There are critical points at the origin and at $y_1 = c$. The origin has characteristic roots c, $-2pc$ so that it is a saddle point. The point A is of Bendixson type. Therefore, as under case (a), there is no s. n. o.

This completes the proof that the system (1.1) possesses no s. n. o. to the right of the y axis.

5. It is much easier to show that (1.1) possesses no s. n. o. to the left of the y axis. To begin with the change of variables $x \longrightarrow -x$, $y \longrightarrow y$ reduces (1.1) to the type

$$\frac{dy}{dx} \doteq \frac{y^2 - 2A(x)y + B(x)}{y - C(x)} \tag{5.1}$$

for which we must show that it has no s. n. o. to the right of the y axis.

Upon drawing the possible sketches for (5.1) it is readily found that Figure 2 is the only one consistent with an s. n. o. The two Γ_H branches are separated by the x axis but the Γ_V branch may run above

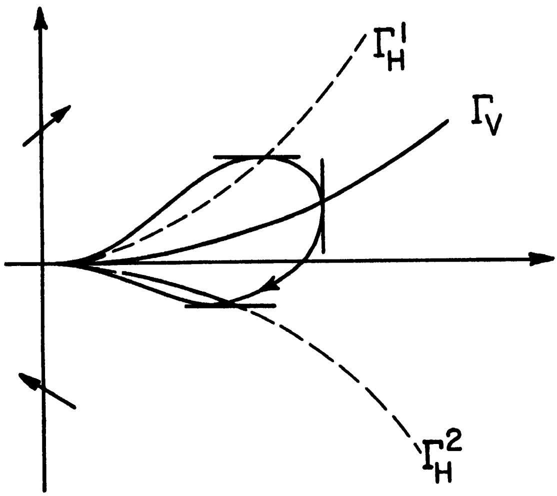

FIGURE 2

or below Ox or even coincide with it. We have again two possible cases:

I. $\quad \Gamma_H^1 : y \doteq ax^p, \quad a > 0$

$\Gamma_H^2 : y \doteq + bx^q, \quad b < 0$

$\Gamma_V : y \doteq cx^r, \quad r > p$ or else $r = p$, and hence $c \neq 0$

II. $\quad \Gamma_H^1 : y \doteq ax^{p + \frac{1}{2}}, \quad a > 0,$

$\Gamma_H^2 : y \doteq - ax^{p + \frac{1}{2}}, \quad p \geqq 1,$

$\Gamma_V : y \doteq cx^r, \quad c \geqq 0, \quad r > p + \frac{1}{2}$.

The justification of the representations for the two Γ_H is as follows. Their general form is

$$y = A(x) \pm \sqrt{\Delta}$$

In order that they be separated by Ox, $\sqrt{\Delta}$ must be of lower order than $A(x)$ and this leads to the indicated representations.

As a matter of fact the change of variables $x^{1/2} \longrightarrow x^*$ reduces II to the type I, so that we only need to consider the latter. Thus (5.1) takes the form

$$\frac{dy}{dx} \doteq \frac{(y-ax^p)(y+bx^q)}{(y-cx^r)} \quad . \tag{5.2}$$

If $q < p$, the transformation $y \longrightarrow -y$ will replace (5.2) by a similar equation with $q > p$. Observe also that $r \geqq p$. Thus we may assume $q \geqq p$. The Newton polygon method gives rise to the following special cases:-

(a) $r = p$. The possible orders are $\mu = p,\ q + 1$. The order $q + 1$ is out since an s. n. o. requires TO-curves of orders $\leqq p$ and $\leqq q$. Thus p is the only possible order. The adequate transformation is $y = x^p y_1$. It yields

$$\frac{dy_1}{dx} \doteq \frac{x(y_1-a)(y_1-bx^{q-p}) + py_1(y_1-c)}{x(y_1-c)} \quad .$$

The only admissible critical point is $y_1 = c$. It is found to be a Bendixson point and hence excluded as before (see 3, Ia).

(b) $r > p$; $q = p + 1$. The only possible order is $\mu = p + 1$. Since one order must be $\leqq p$ this case is ruled out.

(c) $r > p$, $q = p$. The only admissible order is $\mu = p + \frac{1}{2}$ and is ruled out as under (b).

(d) $q > p + 1$. Here we find $\mu = q,\ p + 1$ which excludes $\mu \leqq p$ and so this case is ruled out also.

Thus no s. n. o. can occur to one side of the y axis.

6. There remains to be shown that at most one s. n. o. can be crossed by the y axis.

From (3.1) follows that the polar coordinate equation for $d\theta/dt$ is

$$\frac{d\theta}{dt} \doteq \mp\, y(y - cx^r) \quad .$$

Since $r > 1$ the only possible tangent of the paths at the origin is $y = 0$. Upon drawing one s. n. o. tangent to Ox, it is found to absorb one Γ_V branch (Figure 3). A second set would have to cause the existence of a second such branch. Since there is only one, there can exist at most one set of nested ovals crossed by the y axis. This completes the proof of our theorem.

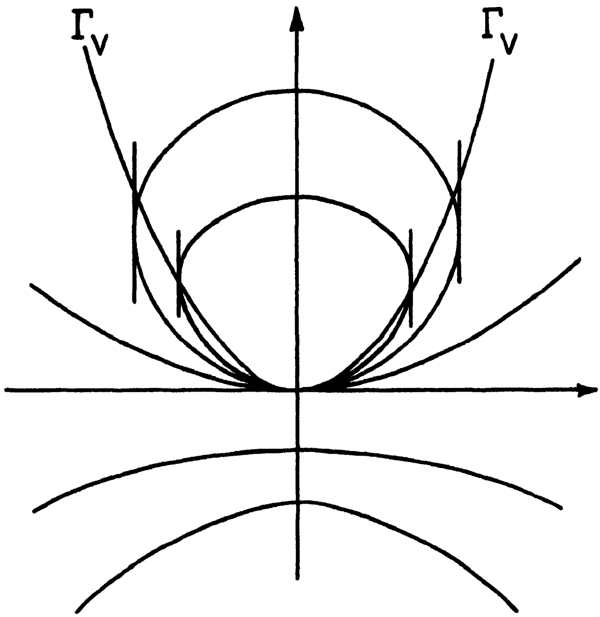

FIGURE 3

That a set of nested ovals corresponding say to Figure 3 can exist is proved by the system

$$\frac{dy}{dx} = \frac{xy}{y - x^2} .$$

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III. OPTIMAL DISCONTINUOUS FORCING TERMS

D. Bushaw

§1. INTRODUCTION

This paper is devoted to a problem which arose originally in the theory of automatic control system design (see [2] or [7]). The problem may be stated in the following form:

<u>If</u> $g(x, y)$ <u>is a given function, find a function</u> $\varphi(x, y)$ <u>with the properties</u>:

(A) $\varphi(x, y)$ assumes only the values -1 and $+1$.

(B) For any point (x_0, y_0) in some plane domain R containing the origin, a solution $x(t)$ of the differential system

$$\ddot{x} + g(x, \dot{x}) = \varphi(x, \dot{x}),\ x(0) = x_0,\ \dot{x}(0) = y_0 \tag{1}$$

exists, and there exists a (least) positive value of t, say t_0, such that for this solution $x(t_0) = \dot{x}(t_0) = 0$.

(C) For all points in R, t_0 is a minimal with respect to the class of functions φ satisfying (A) and (B).

Despite the essentially variational character of this problem, all attempts to apply standard variational methods to it have proved fruitless, and it has been found necessary to develop special approaches which might be described as elaborate combinations of fairly elementary techniques.

The present paper is divided into three principal parts. In the first part (Sections 2 and 3) the problem is recast in a form in which the geometric point of view comes to the fore; in the second part (Section 4) certain general results are obtained which lead to a simplification of the

problem; and in the last and longest part (Sections 5-7) I give the solution of the problem for the case in which the given function g is linear.

§2. TERMINOLOGY

Instead of the differential equation (1), it will be convenient to consider the corresponding "phase" system

$$(2) \qquad \dot{x} = y, \ \dot{y} = \varphi(x, y) - g(x, y) \ .$$

I shall assume throughout that the given function $g(x, y)$ has continuous first partial derivatives everywhere.

Suppose for the moment that the function $\varphi(x, y)$ is identically $+ 1$; then the solution curves of (2) cover the entire plane exactly once. This family of curves will be called the <u>P-system</u>, its curves <u>P-curves</u>, and the arcs of its curves <u>P-arcs</u>. Likewise, when $\varphi \equiv - 1$, one obtains a family of solution curves which will be called <u>N-curves</u>, etc. Each P- or N-arc is automatically oriented by the increase of t along it.

There is some question about what should be meant by a solution of (2) when φ takes on both of the values $- 1$ and $+ 1$, and is therefore discontinuous. In order to avoid ambiguities, I shall adopt the following definition which, although it involves an element of arbitrariness, accords well with the classical definition and with the physical context in which the problem arises. For the sake of simplicity, the definition will be expressed in geometric language.

Suppose that $p_0 = (x_0, y_0)$ is the point in the plane from which the solution is sought, and suppose $\varphi(x_0, y_0) = + 1$. Then one of the three following mutually exclusive possibilities must be realized:

(i) There exists a P-arc beginning at p_0 of positive length along which $\varphi(x, y) = + 1$.

(ii) The condition (i) is not satisfied, but there exists an N-arc from p_0 along which $\varphi(x, y) = - 1$ at every point but p_0.

(iii) Neither (i) nor (ii) holds.

If (i), then if there exists a first point after p_0 on the P-curve from p_0 at which $\varphi(x, y)$ changes sign, the solution is defined to begin with the P-arc from p_0 to this point p_1. If there exists no such point, the solution from p_0 is defined to be that part of the P-curve through p_0 which follows p_0 (the <u>P-semicurve</u> from p_0).

If (ii), the preceding paragraph should be applied with N in place of P.

If (iii), no solution from p_0 is defined.

Cases (i) and (ii) define the solution from p_0 wholly or up to a certain point p_1. In the latter case, the above process should be repeated with p_1 in place of p_0, the letters P and N interchanged, and the numbers -1 and $+1$ interchanged. This will lead to the same trichotomy: either the solution is not defined beyond p_1, or it is completed by a whole N- or P-semicurve originating at p_1, or it is defined up to a certain point p_2. Then p_2 (if it arises) is to be treated as p_0 was, and so on.

If $\varphi(x_0, y_0) = -1$, the solution from p_0 is defined analogously.

Thus a solution of (2) consists of a countable (possibly finite or even vacuous) well-ordered sequence of alternating P- and N-arcs such that the initial point of the first arc is p_0, the terminal point of each arc is the initial point of the next, and $\varphi(x, y) = +1$ on the P-arcs, -1 on the N-arcs.

If Δ is the solution curve of (2) starting at a point p, and p' is any point on Δ, then the solution curve starting at p' is that part of Δ which follows p. Moreover, a solution curve cannot intersect itself at a point p unless it is periodic beyond p. (Here a solution can be "periodic beyond a point" without being completely periodic because the solutions are defined only unilaterally.)

A point on a solution curve which is the terminal point of a P-arc and the initial point of an N-arc will be called a <u>PN-corner</u>. <u>NP-corners</u> are defined analogously.

§3. PATHS

Instead of dealing with $\varphi(x, y)$ directly, we shall deal with curves which might possibly occur as solution curves of (2) for various choices of φ, and consider the problem of finding such a curve of least possible time length which connects a given point p with the origin. It will be shown below that solving this problem is tantamount to solving the original problem.

It is assumed throughout this section and the next that the function g, and therefore the P- and N-systems, are given.

A <u>path</u> from the point p is a countable, well-ordered sequence of alternating P- and N-arcs such that:

1) The sum of the time lengths of the arcs is finite $(= \tau)$.

2) The initial point of the first arc is p.

3) The terminal point of each arc is the initial point of the next.

4) If there are finitely many arcs, the terminal point of the last arc is the origin; if there are infinitely many, then $x(t) \longrightarrow 0$ and $y(t) \longrightarrow 0$ as $t \longrightarrow \tau$, the path being parametrized in the obvious way in terms of the parametrization of the constituent arcs.

5) No two of the arcs intersect.

In order to avoid a conflict between 3) and 5), each arc is to be regarded as containing its initial point but not its terminal point; this convention will not restrict the generality of what follows. A path from p can therefore almost be described as a curve which could occur as that part of a solution from p (for some φ) which connects p with the origin. "Almost," because 5) need not hold for every solution of (2); but since our object is to find solution curves of shortest possible time length, nothing will be lost if we leave self-intersecting solutions out of consideration.

A path from p whose time length τ is not longer than that of any other path from p will be called a <u>minimal path</u> from p.

> <u>In order to solve the problem stated in Section 1 it is sufficient to prove that there exists a unique minimal path from each point p in some domain R containing the origin</u>.

Namely, one needs only to define $\varphi(x, y) = +1$ on P-arcs which occur in the minimal paths, and $\varphi(x, y) = -1$ on N-arcs which occur in the minimal paths. Such a $\varphi(x, y)$ automatically yields the minimal paths as solution curves, and the minimal path from a point p is, by definition, the solution curve of least possible time length connecting p with the origin.

This method defines $\varphi(x, y)$ uniquely at every point of R except the origin, where the value of φ doesn't matter. To verify this, observe first that every point p of R must lie on at least one minimal path, the minimal path which begins at p. Thus φ is given a value at every point of R. This value is unique, for if there were a point p at which it failed to be unique, this point would necessarily lie both on an N-arc belonging to one minimal path Δ_1 (say from the point p_1) and on a P-arc belonging to another minimal path Δ_2 (from p_2). Denote those parts of Δ_1 and Δ_2 which lie between p and the origin by $\tilde{\Delta}_1$ and $\tilde{\Delta}_2$ respectively. Their time lengths $\tau(\tilde{\Delta}_1)$ and $\tau(\tilde{\Delta}_2)$ stand in some relation to each other, say $\tau(\tilde{\Delta}_1) \leq \tau(\tilde{\Delta}_2)$; then $\tau(\Delta_2 - \tilde{\Delta}_2 + \tilde{\Delta}_1) \leq \tau(\Delta_2)$.

The curve $\Delta_2 - \tilde{\Delta}_2 + \tilde{\Delta}_1$ may not be a true path (for it may cross itself), but a true path may be obtained from it by cutting out whatever closed loops or retracings it may contain; and the resulting path is clearly shorter in time length than Δ_2, but this contradicts the assumption that Δ_2 was the unique minimal path from p_2.

§4. CANONICAL PATHS

A path will be called <u>canonical</u> if it contains no NP-corners above the x-axis and no PN-corners below it. In saying that a corner lies above or below the x-axis, I mean that nearby parts of the arcs meeting at the corner are above or below it; the corner itself, regarded as a point, may lie <u>on</u> the axis.

> THEOREM 1. Given any path Δ from p which is not canonical, one can find a canonical path from p whose time length is less than that of Δ.

The idea of the proof is simple: given, say, a path with the NP-corner p above the x-axis, one denotes by p' either the last corner of the path preceding p or the last intersection preceding p of the path with the x-axis (whichever is nearer to p), and denotes by p'' the corresponding point following p on the path. One then draws the P-curve forward from p' and the N-curve backward from p'', thereby obtaining a four-sided figure as shown (Figure 1). If one now modifies the given path

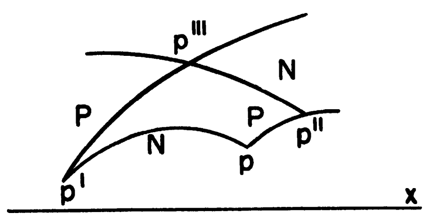

FIGURE 1

by replacing the section $p'pp''$ by the section $p'p'''p''$, the NP-corner is removed, no other such corner is introduced, and the time length of the path is reduced. To see this last fact, note that by (2)

$$\tau(p'pp'') = \int_{p'pp''} y^{-1}dx, \quad \tau(p'p'''p'') = \int_{p'p'''p''} y^{-1}dx \;,$$

where the integrals are extended over the indicated curves. However, for any given value of x the value of y is greater on $p'p'''p''$ than on $p'pp''$; therefore the second integral must be smaller than the first, as claimed. If one applies this process to every NP-corner above the x-axis, and the corresponding process to every PN-corner below the axis, the result is a canonical path shorter than the given path. To complete the justification of this conclusion, one must verify two things: (A) that it is always possible to construct the "quadrilateral" shown in Figure 1; and (B) that the process described introduces no self-crossings, so that a true path is in fact obtained. The assertion (B) may be established by a straightforward survey of the possibilities, the fact that P- and N-curves can cross only in certain ways being taken into account. The verification of (A) follows.

Let p, p', and p'' be as described above; if the initial point of a path is regarded as a corner, p' always exists, and the existence of p'' follows from the fact that the path goes to the origin. It will be shown first that the P-semicurve Π beginning at p' passes over $p'pp''$ and crosses the vertical line through p''. That Π moves to the right as long as it remains above the x-axis follows from the first equation in (2). If Π is given by the functions $\bar{x}(t)$, $\bar{y}(t)$, where $p' = (\bar{x}(0), \bar{y}(0))$, then Π has one of the following three properties:

(i) For some $t_0 > 0$, $\bar{y}(t_0) = 0$.

(ii) $\liminf_{t \to \infty} \bar{y}(t) = 0$.

(iii) $\bar{x}(t) \longrightarrow \infty$ as $t \longrightarrow \infty$.

For if both (i) and (ii) are false, there exists a number $\epsilon > 0$ and a value $T > 0$ of t such that for $t > T$, $\bar{y}(t) > \epsilon$. Since $\dot{\bar{x}} = \bar{y}$, this gives

$$\bar{x}(t) - \bar{x}(T) = \int_T^t \bar{y}(t')dt' > \epsilon\,(t - T) \longrightarrow \infty \quad \text{as} \quad t \longrightarrow \infty,$$

so that (iii) holds.

Thus, as t increases Π must either approach the x-axis as closely as one wishes or move off infinitely far to the right.

Π starts off from p' above $p'pp''$, for in the upper half-plane the P-curve through a point always has a greater slope there than the N-curve through that point, and even if p' lies on the x-axis (so that both curves have vertical tangents at p'), the radius of curvature of the P-curve at p' is greater than that of the N curve. This all follows from (2). Π cannot cross the N-arc $p'p$, by what was just said about slopes;

it cannot cross the P-arc pp'', since P-curves do not intersect one another; and it cannot tend as $t \longrightarrow \infty$ to either of the points p or p'' (which may be on the x-axis), for this would imply that the point concerned would be a singular point of the P-system, and this in turn would be inconsistent with the fact that the time length of pp'' is finite. Thus all that was claimed for Π is true.

The corresponding argument can be given for N, the N-semicurve obtained by following the N-curve through p'' backwards from p'': N also lies above $p'pp''$, and crosses the vertical line through p'. Thus Π and N must intersect at least once; but they can intersect only once, for if there were two consecutive intersections one of them would necessarily involve an impossible inequality between the slopes. This gives the unique point p''' and the "quadrilateral" sought.

This completes the proof of Theorem 1.

COROLLARY. In seeking a minimal path from a point it is sufficient to consider only canonical paths from that point.

For it follows immediately from Theorem 1 that a path which is minimal with respect to the class of all canonical paths is also minimal with respect to the class of all paths whatever. <u>From this point on all paths considered are therefore assumed to be canonical</u>.

If $g(x, y)$ happens to have the property $g(-x, -y) = -g(x, y)$, discussions can be simplified; for if on this hypothesis we make in the equations (2) the substitutions $x = -X$, $y = -Y$, $\varphi(x, y) = -\Phi(X, Y)$, we obtain

$$\dot{X} = Y, \quad \dot{Y} = \Phi(X, Y) - g(X, Y) \quad ,$$

and these equations have exactly the same form as (2). This means that if p and q are two points symmetrical with respect to the origin, then whatever can be said about the P- and N-curves at p can be said about the N- and P-curves at q: e.g., if it can be proved that any minimal path from p must begin with a P-arc, it follows at once that a minimal path from q must begin with an N-arc. Since $g(x, y)$ always has the property named when it is linear (homogeneous), and $g(x, y)$ will be linear in subsequent sections, this observation will be extensively used. Any result obtained from another by an appeal to it will be said to have been obtained <u>by symmetry</u>.

§5. THE LINEAR CASE WITH COMPLEX CHARACTERISTIC ROOTS

In the rest of this paper the fundamental problem of Section 1 will be restricted by supposing that $g(x, y) = Ax + By$, where A and B are (real) constants. It is well known that the qualitative behavior of the P- and N-curves will then depend on the nature of the characteristic roots of the matrix

$$\begin{pmatrix} 0 & 1 \\ -A & -B \end{pmatrix} .$$

The first case considered here is that in which these characteristic roots are <u>complex</u>. In this case (after a suitable redefinition of the units in which x and t are measured, if necessary) the equations (2) become

$$(3) \qquad \dot{x} = y, \quad \dot{y} = \varphi(x, y) - (x + 2by) \quad ,$$

where $|b| < 1$. In this section it will be assumed further that $b \geq 0$.

When $b > 0$ the P-curves for (3) are clockwise-oriented spirals for which the point $(1, 0)$ is the (stable) focus. If $b = 0$ the P-curves are ellipses with $(1, 0)$ as the center. The N-system may be obtained in either case by translating the P-system two units to the left along the x-axis.

Let

$$\alpha = \sqrt{1 - b^2}$$

and consider the transformation

$$(4) \qquad X = x + by, \quad Y = \alpha y \quad .$$

The use of X and Y may be regarded as the use of oblique coordinates in the (x,y)-plane. Under (4), the system of equations (3) becomes

$$(5) \qquad X = - bX + \alpha Y + b\varphi, \quad Y = - \alpha X - bY + \alpha\varphi \quad .$$

The P- and N-curves are given explicitly by

$$(6) \qquad \begin{aligned} X(t) &= e^{-bt}(Ae^{i\alpha t} + Be^{-i\alpha t}) \pm 1 \quad , \\ Y(t) &= ie^{-bt}(Ae^{i\alpha t} - Be^{-i\alpha t}) \quad , \end{aligned}$$

where

$$(7) \qquad A = \frac{1}{2}\,[X(0) - iY(0) \mp 1], \quad B = \bar{A} \quad .$$

(Here and throughout what follows the upper sign pertains to the P-system, the lower to the N-system. The functions (6) represent ordinary logarithmic spirals or, if $b = 0$, circles.)

LEMMA 1. The time length of any arc cut off from a P- or N-curve by two successive intersections with the X-axis is π/α.

This may be easily verified by considering (6) and (7).

A useful sequence of numbers ξ_n is defined as follows: First, put $\xi_0 = 0$. The P-curve starting at $(\xi_0, 0) = (0, 0)$ reaches, after the lapse of $-\pi/\alpha$ time units, a point $(\xi_1, 0)$; the N-curve starting at $(\xi_1, 0)$ reaches, after the lapse of $-\pi/\alpha$ time units, a point $(\xi_2, 0)$; and so on.

LEMMA 2. The values of the numbers ξ_n are given by

$$\xi_n = (-1)^n\left(e^{nb\pi/\alpha} + 1 + 2\sum_1^{n-1} e^{kb\pi/\alpha}\right) \qquad (n = 1, 2, \ldots)$$

This formula may be derived from equations (6) and (7) by complete induction.

The letter Γ will be used in this section to denote the P-arc joining the point $(\xi_1, 0)$ with the origin.

LEMMA 3. If $p \in R$, where R is the set which consists of the interval $0 < X < \xi_1$ on the X-axis and the interior of the region bounded by this interval and Γ, and if Δ is that path from p which is obtained by following the N-curve through p to Γ and then following Γ into the origin, $\tau(\Delta) < \pi/\alpha$.

PROOF. It will suffice to prove this for points on the X-axis only. The equations of Γ are

$$(8) \qquad \begin{aligned} X(t) &= 1 - \tfrac{1}{2}e^{-bt}(e^{i\alpha t} + e^{-i\alpha t}), \\ Y(t) &= \tfrac{1}{2}ie^{-bt}(e^{-i\alpha t} - e^{i\alpha t}), \end{aligned}$$

where $-\pi/\alpha \leq t \leq 0$. The point $q = (X_q, Y_q)$ is given by these equations when t assumes some value $-\lambda(0 < \lambda < \pi/\alpha)$. If q is regarded as the initial point of the N-arc pq, the time necessary to reach p being $-\mu(0 < \mu < \pi/\alpha)$, one obtains

$$(9)\qquad \begin{array}{c} e^{b\mu}(Ae^{-i\alpha\mu} + Be^{i\alpha\mu}) + 1 \\ ie^{b\mu}(Ae^{-i\alpha\mu} - Be^{i\alpha\mu})\ [= 0] \end{array}$$

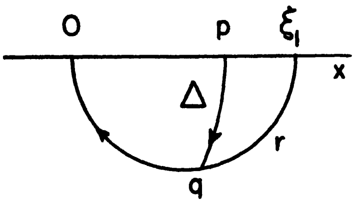

FIGURE 2

as the coordinates of p, where $A = \bar{B} = \frac{1}{2}(X_q + 1 - iY_q)$. The fact that the quantities (9) are real implies that $Ae^{-i\alpha\mu}$ is real. Putting its imaginary part equal to zero and using the explicit expressions for X_q and Y_q one gets

$$e^{b\lambda}\sin\alpha(\lambda + \mu) = 2\sin\alpha\mu \quad .$$

Because $0 < \alpha\mu < \pi$, the right side, and therefore the left side of this equation is positive. This, together with the inequality $0 < \alpha(\lambda + \mu) < 2\pi$, implies $0 < \alpha(\lambda + \mu) < \pi$, which was to be shown.

With the aid of these three lemmas one can start finding minimal paths. The procedure will be to examine one part of the lower half-plane after another until a unique minimal path from each point in this region has been found; the rest will follow by symmetry.

THEOREM 2. If p is any point on the interval $0 < X < 1$ of the X-axis, then given any path Δ from p which begins with a P-arc and whose time length is less than π/α, one can find a path from p which begins with an N-arc and whose time length is less than that of Δ.

PROOF. Δ must behave as follows: it begins (by assumption) with a certain P-arc pq. This cannot return to the X-axis, for if it did it alone (and therefore the whole path) would have a time length exceeding π/α, by Lemma 1. From q, Δ follows some N-arc which crosses the X-axis but, as before, cannot cross it again. (It does cross the X-axis because if it were to stop short of the X-axis, or on the interval $0 < X < 1$, the path would not be canonical; and if it were to stop on the X-axis to the right of $(1, 0)$, the succeeding P-arc would necessarily return to the X-axis and thus be too long.) The

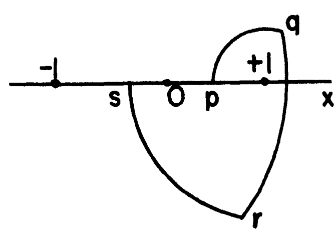

FIGURE 3

N-arc qr is followed by a P-arc which likewise must cross the axis at some point s, which must lie to the left of p, since otherwise Δ could not reach the origin from s without crossing itself. How Δ behaves beyond s is irrelevant.

If $\lambda = \tau(pq)$, $\mu = \tau(qr)$, and $\sigma = \tau(rs)$, one can repeatedly apply (6) and (7) so as to obtain the coordinates of q, r and s in terms of these three variables and the abscissa of p; and from the resulting expressions elimination gives

$$(10) \qquad X_s - 1 = e^{-\gamma\sigma}\left\{\left[(X_p - 1)e^{-\gamma\lambda} + 2\right]e^{-\gamma\mu} - 2\right\}, \qquad \gamma = b - \alpha i.$$

It will be shown that if p and s are held fixed and λ is reduced to 0, the result is a path which is shorter than Δ. First, $d(\lambda + \mu + \sigma)/d\lambda > 0$; for since p and s are to be held fixed, λ can be taken as the sole independent variable in (10); and differentiating both sides of (10) with respect to λ leads to

$$(11) \qquad \frac{d}{d\lambda}(\lambda + \mu + \sigma) = \frac{2e^{\gamma\sigma}(d\sigma/d\lambda) + 1}{(X_p - 1)e^{-\gamma\lambda} + 2} \quad .$$

This quantity is of course real; putting its imaginary part equal to zero gives

$$(12) \qquad \begin{aligned} \frac{d\sigma}{d\lambda}\, e^{b\mu}\left[2 \sin \alpha\mu + (X_p - 1)e^{-b\lambda}\sin \alpha(\lambda + \mu)\right] \\ = (1 - X_p)e^{-b\lambda}\sin \alpha\lambda \quad . \end{aligned}$$

Using this to eliminate $d\sigma/d\lambda$ from the right member of (11) gives

$$\frac{d}{d\lambda}(\lambda + \mu + \sigma) = \frac{K \sin \alpha\mu}{2 \sin \alpha\mu + (X_p - 1)e^{-b\lambda}\sin \alpha(\lambda + \mu)} \quad ,$$

where K is a quantity which can be seen to be positive on purely algebraic grounds. It follows from (12) that the above denominator is positive; thus the derivative has the sign of $\sin \alpha\mu$. When λ has its original value, this is positive, since $0 < \lambda + \mu + \sigma < \pi/\alpha$. As λ is decreased the whole sum $\lambda + \mu + \sigma$ decreases, μ remains less than π/α, and the derivative remains positive. Thus we may shorten Δ by decreasing λ; and λ may be decreased without changing the topology of the situation until one of two things happens:

I. r <u>comes into coincidence with</u> s. But this is impossible, for if it occurred the shortened path would then contain an N-arc $q's$

which would intersect the X-axis twice and thus be impossibly long.

II. q comes into coincidence with p; that is, λ goes all the way to zero. This leads to the path whose existence was asserted.

COROLLARY. If p is any point of the region R (see Lemma 3), the conclusion of Theorem 2 remains true.

PROOF. It is easy to see, by arguments like those used above, that a sufficiently short path from p beginning with a P-arc must be of the type pp'qrs ... shown in Figure 4. From the proof of the theorem it

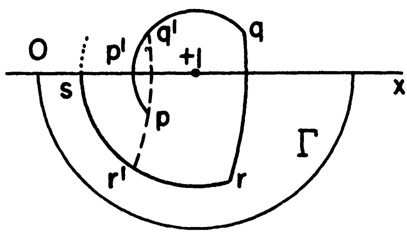

FIGURE 4

follows that decreasing $\lambda = \tau(p'q)$ reduces the time length of the path. This process may be continued until either I (as described above) occurs -- but this is impossible for the same reason as before -- or the arc qr comes to contain the point p. In this case cutting off the closed loop pp'q'p leaves the path whose existence was to be proved.

COROLLARY. If $p \in R$, and Δ is any given path from p such that $\tau(\Delta) < \pi/\alpha$ and which does not begin with an N-arc intersecting Γ, there exists a path from p which does and whose time length is less than that of Δ.

PROOF. Let q be the initial point of the first P-arc of Δ. By assumption, $q \in R$. The preceding corollary states that we may reduce the time length of Δ be replacing q ... with a path from q which begins with a non-vacuous N-arc qq_1. If $q_1 \notin R$, there is nothing more to do; if $q_1 \in R$, the procedure is to be repeated. Sooner or later the point q_n, the initial point of the first P-arc of the path after n such modifications, must lie on or below Γ; for each point q_n is like r'

(Figure 4) and if all the points q_n were to lie in R, then all the corresponding points like s on Δ would lie to the right of the origin, and this is inconsistent with the fact that Δ reaches the origin.

THEOREM 3. If $p \in R$, the unique minimal path from p is obtained by following the N-curve through p to Γ and then following Γ into the origin. (If $p \in \Gamma$, the unique minimal path from p is obtained simply by following Γ into the origin.)

PROOF. By Lemma 3, we know that there exists at least one path from p the time length of which is less than π/α; so we need consider only such paths. Then, by the second corollary to Theorem 2, we can further restrict our attention to those paths from p which follow the N-curve from p at least until it reaches Γ. Furthermore, it is enough to consider points p of R which lie on the X-axis; for once the theorem is proved for this case, it automatically follows for all points on the N-arcs connecting such points p with Γ.

Let Δ be a path meeting these conditions. It begins with an N-arc which crosses Γ but does not return to the X-axis. The corner q is followed by a P-arc qr which must cross the X-axis once (it cannot stop there -- unless q = 0 -- for this would force the succeeding N-arc to be too long) but, for the usual reason, does not return to it. The N-arc of Δ beginning at r intersects the X-axis at some point s. The rest of the proof follows that of Theorem 2 almost to the letter. The time length of pqrs is obtained as a function of that of the N-arc pq, and by differentiation is shown to be an increasing function of that variable. Thus the time length of Δ may be reduced by holding s fixed and reducing the length of the arc pq to its least possible value, i.e., until $q \in \Gamma$. The reduced "path" then reaches the origin before it reaches its next corner after q (in its new position), and is therefore still at least as long timewise as the path which the theorem asserts to be minimal. Thus the given path Δ is of greater time length than the asserted minimal path, q.e.d.

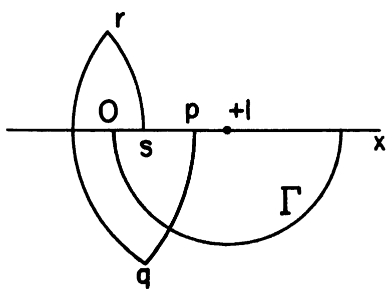

FIGURE 5

Theorem 3 constitutes the first step in an inductive argument the whole of which will give the complete solution of the problem.

THEOREM 4. If p lies on the interval $|\xi_n| < X \leq |\xi_{n+1}|$ of the X-axis, then the unique minimal path from p is that which consists of $n + 2$ arcs, the first of which is an N-arc of length $\lambda (0 \leq \lambda < \pi/\alpha)$, the last of which is of length $\sigma (0 < \sigma \leq \pi/\alpha)$, and the intervening ones of which are all of length π/α.

The proof will be by induction on n $(n = 0, 1, \ldots)$. Theorem 3 gives the desired result for $n = 0$. The first step is to determine the locus of the corners belonging to the paths described in the theorem. This will show incidentally that paths meeting the specifications of the theorem exist and are unique. It is clear that the corners must be obtainable in the following way: working backwards from the origin, one follows Γ (or Γ^-, the reflection of Γ through the origin, depending on the parity of n) for some time interval $-\sigma (0 < \sigma \leq \pi/\alpha)$, then turns onto an N-arc (resp. P-arc) and follows it for the time interval $-\pi/\alpha$, thence follows a P-arc (resp. N-arc) for the same length of time, and continues to follow alternating P- and N-arcs until n of them have been traversed. The ends of the n^{th} arcs, as σ ranges over its interval, describe a certain curve E_n which is the locus of the first corners on the paths described in the theorem. It remains to verify that E_n is in the right place. By complete induction applied to the formulas (6) and (7), one may obtain as parametric equations of E_n

$$X_n(\sigma) = e^{nb\pi/\alpha}(1 - e^{b\sigma}\cos \alpha\sigma) + |\xi_n| \quad ,$$

(13)

$$Y_n(\sigma) = e^{nb\pi/\alpha}(- e^{b\sigma}\sin \alpha\sigma) \quad ,$$

where $0 < \sigma \leq \pi/\alpha$. E_0 is merely Γ; but from (13) it appears that E_n is the curve obtained by magnifying Γ by the factor $e^{nb\pi/\alpha}$ and then translating the result $|\xi_n|$ units to the right. For each n, the curves E_n and E_{n+1} meet in a cusp at the point $(|\xi_{n+1}|, 0)$. From these considerations it is easy to see that the paths asserted to be minimal in the theorem exist and are unique; moreover, their corners are all on the curves E_n and those obtained from the curves E_n by reflection through the origin.

We are now prepared to embark on the proof proper of Theorem 4. As in previous cases, one begins by deciding how a path from p $(p = (X_p, 0), |\xi_n| < X_1 \leq |\xi_{n+1}|)$ which might be minimal can behave. It begins, let us say, with an N-arc of time length λ, where $\lambda \geq 0$. We shall <u>assume</u> for the present that λ also satisfies the condition

(14) $$\lambda \leq \pi/\alpha \quad ;$$

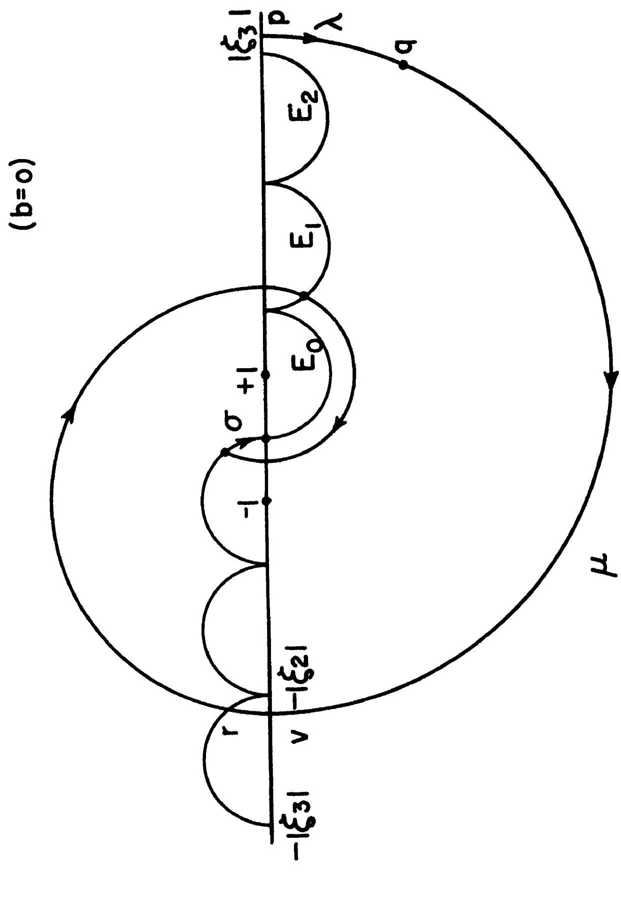

FIGURE 6

this assumption will be justified below. Thus the first arc of the path ends at a point q on or below the X-axis, and from q the second arc qr, a P-arc, emerges. Because the path is canonical, the arc qr must cross the X-axis, say at the point v; but here again we make a heavier assumption for which justification will be supplied later:

(15) The point v lies on the interval
$$-|\xi_n| \leq X < -|\xi_{n-1}| \quad .$$

We can now assume (and this requires no further justification) that after the path passes v, it coincides with the minimal path from v given by the inductive assumption and symmetry. Thus r must have the coordinates (cf. (13))

$$X_r = e^{(n-1)b\pi/\alpha}(e^{b\sigma}\cos\alpha\sigma - 1) - |\xi_{n-1}| \quad ,$$

$$Y_r = e^{(n-1)b\pi/\alpha}e^{b\sigma}\sin\alpha\sigma \quad ,$$

where σ, the time length of the last arc of the path, satisfies $0 < \sigma \leq \pi/\alpha$ and depends only on p and λ, the time length of pq. If the length of qr is μ, then using (6) and (7) for the P-arc of time length $-\mu$ with the initial point r one can calculate the coordinates of q as functions of σ and μ. Then following the N-curve from q for the time length $-\lambda$, one gets the coordinates of p as functions of λ, μ, and σ. With the use of Lemma 2 the result of this calculation can be put in the form

$$X_p + 1 = e^{\gamma\lambda}\left[\left(\rho^{n-1}e^{\gamma\sigma} - 2\sum_{0}^{n-1}\rho^k\right)e^{\gamma\mu} + 2\right], \tag{16}$$

where $\rho = e^{b\pi/\alpha}$ and again $\gamma = b - \alpha i$. Our problem is to minimize the quantity $T = \lambda + \mu + \sigma + (n-1)\pi/\alpha$, the time length of the path, under the conditions set out above. For a fixed p, T may be regarded as a function of λ. The range of variation of λ is $0 \leq \lambda \leq \lambda_0$, where λ_0 is that value of λ corresponding to $\sigma = \pi/\alpha$. The remainder of the proof follows familiar lines. The result of differentiating both sides of (16) and suitably rearranging the terms is

$$\frac{dT}{d\lambda} = 2\,\frac{(\Sigma_0^{n-1}\rho^k)(d\sigma/d\lambda) + e^{-\gamma\mu}}{2(\Sigma_0^{n-1}\rho^k) - \rho^{n-1}e^{\gamma\sigma}} \tag{17}$$

The denominator cannot vanish for the values of b and σ which have been admitted. Separating real and imaginary parts on the right side of (17), setting the latter equal to zero, and using the resulting equation to eliminate $d\sigma/d\lambda$ from (17) yields

$$dT/d\lambda = K \sin \alpha\mu / \sin \alpha\sigma \quad ,$$

where K is positive in any case. Since $0 < \sigma \leq \pi/\alpha$, the sign of the derivative is that of $\sin \alpha\mu$. However, $\sin \alpha\mu = 0$ if q is on E_n. When λ is smaller, so that q is above E_n, $\sin \alpha\mu$ and therefore $dT/d\lambda$ are negative; when larger, positive. Thus T assumes its minimum value when $\mu = \pi/\alpha$ $(q \in E_n)$, as was to be shown. (In the extreme case $X_p = |\xi_{n+1}|$, the only path meeting our conditions is given by $\lambda = 0$ and $\mu = \pi/\alpha$; thus it must be minimal.)

It remains to justify the assumptions (14) and (15) used above. The assumption (14) will be justified by showing that a path from p which satisfies (14) can be found with a time length shorter than that of any prescribed path from p which does not satisfy (14). To say that (14) does not hold for a path Δ is to say that the initial N-arc of Δ not only returns to the X-axis (on the negative half) but crosses it. Let v be the point at which this happens. If one by-passes the point v by means of a short P-arc, a canonical path which satisfies (14) and has a shorter time length than Δ is obtained. That this is possible follows from the fact that at v the P- and N-curves have a common (vertical) tangent, while the curvature of the P-curve is less than that of the N-curve; this may be verified using the underlying differential equations. If the P-arc introduced is short enough, it cannot touch Δ at any points but its endpoints, so the modified path is again canonical. That it satisfies (14) should be obvious; that its time length is shorter than that of Δ follows from the proposition:

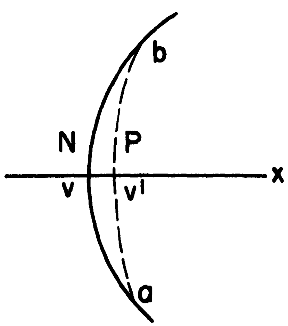

FIGURE 7

If $p = (\xi, \eta)$, *where* $\xi < -1$, *is a point near the X-axis, and if* λ *and* λ' *are the time lengths of the respective shortest P- and N-arcs joining* p *with the X-axis, then* $\lambda < \lambda'$.

Proof: Let $\eta < 0$, and regard p as the initial point of the two

arcs. Using (6) and (7), together with the fact that the end points of the arcs are on the X-axis, one finds that the quantities

$$[(\xi - 1) - i\eta]e^{i\alpha\lambda}, \quad [(\xi + 1) - i\eta]e^{i\alpha\lambda'}$$

must be real. Setting their imaginary parts equal to zero gives

$$\tan \alpha\lambda = - \eta/(- \xi + 1), \quad \tan \alpha\lambda' = - \eta/(- \xi - 1) \ .$$

The desired inequality now follows from the fact that $\tan u$ is an increasing function near $u = 0$, since $- \eta/(- \xi + 1) < -\eta/(- \xi - 1)$.

In the following justification of (15) it will accordingly be assumed that all paths considered satisfy (14). It will be shown that any such path for which (15) is not true must have a greater time length than $\tilde{\Delta}_p$, the path from p which the theorem asserts to be minimal.

Let Δ be such a path. It begins with an N-arc of length $\lambda (0 \leq \lambda \leq \pi/\alpha)$. If the terminal point of this arc is again called q, then q is the initial point of a P-arc on Δ which, by the argument just given and symmetry, may be supposed to stop short of crossing the positive half of the X-axis. If r is the terminal point of this arc, there comes next an N-arc starting at r which goes at least to the positive half of the X-axis, say at the point s. It follows from the definition of ξ_n that $|\xi_{n-1}| < X_s$. On the other hand, since Δ does not cross itself, it must be that

$$X_s < X_p < |\xi_{n+1}|$$

We cannot be so specific about what Δ does beyond s; but sooner or later, in order to be able to reach the origin, Δ must cross one or the other of the intervals

$$(18) \qquad |\xi_{n-1}| < X \leq |\xi_n|, \quad - |\xi_n| \leq X < - |\xi_{n-1}| \ ,$$

for these intervals together with the N-arc joining $(- |\xi_n|, 0)$ with $(|\xi_{n-1}|, 0)$ and the P-arc joining $(|\xi_n|, 0)$ with $(- |\xi_{n-1}|, 0)$, neither of which can be crossed by Δ, form a simple closed curve separating p from the origin. Let $N(\Delta)$ denote the number of times Δ crosses the X-axis before crossing one of the intervals (18); our problem is to show that if $N(\Delta) > 0$, $\tau(\Delta) > \tau(\tilde{\Delta}_p)$. To this end we first prove:

$$(19) \qquad n\pi/\alpha < \tau(\tilde{\Delta}_p) \leq (n + 1)\pi/\alpha \ .$$

In terms of the notation developed in the main part of the proof, we know that

$$\tau(\tilde{\Delta}_p) = \lambda + \sigma + n\pi/\alpha \quad .$$

Putting $\mu = \pi/\alpha$ in (16), one obtains:

$$X_p + 1 = e^{\gamma\lambda}[2\Sigma_0^n \rho^k - \rho^n e^{\gamma\sigma}] \quad .$$

Since the imaginary part of the right member vanishes, we have

$$\rho^n e^{b\sigma} \sin \alpha(\lambda + \sigma) = 2(\Sigma_0^n \rho^k) \sin \alpha\lambda \quad .$$

But on $\tilde{\Delta}_p$, $0 \leq \lambda < \pi/\alpha$ and $0 < \lambda + \sigma < 2\pi/\alpha$. These inequalities and the above equation imply $0 < \lambda + \sigma \leq \pi/\alpha$, and this in turn implies (19).

We can say first that if $N(\Delta) \geq 3$, then Δ has a greater time length than $\tilde{\Delta}_p$; for when $N(\Delta) \geq 3$, Δ begins with a curve pqrs as described above, and s is followed by a curve sq'r's' of the same sort. If v and v' are the points at which the arcs qr and q'r' intersect the X-axis, then one can use methods already developed to show that each of the four pieces pqv, vrs, sq'v', and v'r's' has a time length greater than $\pi/2\alpha$; and since the time length of that part of Δ following s' is, by the inductive assumption and (19), greater than $(n - 1)\pi/\alpha$, the time length of Δ itself must be greater than $(n + 1)\pi/\alpha$, which, by (19), is in turn greater than $\tau(\tilde{\Delta}_p)$.

The remaining cases, $N(\Delta) = 2$ and $N(\Delta) = 1$, may be settled individually by use of the method of path contraction already used so extensively above; the details, which are novel only in minor respects, will therefore be omitted from this account.

This completes the proof of Theorem 4.

COROLLARY. If p lies in the closed region bounded by one of the curves E_n and the X-axis, the unique minimal path from p is that obtained by following the path described in Theorem 4 which passes through p.

PROOF. Suppose that Δ_p were a path from p such that $\tau(\Delta_p) \leq \tau(\tilde{\Delta}_p)$, where $\tilde{\Delta}_p$ is that path from p which the corollary asserts to be minimal. Let p' be that point on the X-axis which is first reached by following the N-curve through p backwards; clearly, $|\xi_n| \leq X_{p'} \leq |\xi_{n+1}|$. The curve composed of the N-arc p'p and Δ_p will be (after the

elimination of any loops, retracings, etc., that it may happen to contain) a certain path from p', say $\Delta_{p'}$. But

$$\tau(\Delta_{p'}) \leq \tau(p'p) + \tau(\Delta_p) \leq \tau(p'p) + \tau(\tilde{\Delta}_p) = \tau(\tilde{\Delta}_{p'}) \quad ,$$

and this contradicts the fact (Theorem 4) that $\tilde{\Delta}_{p'}$ is the unique minimal path from p'.

Since $b \geq 0$, $|\xi_n| \longrightarrow \infty$ as $n \longrightarrow \infty$, and the minimal paths discovered by Theorem 4 sweep out the entire plane; and by a simple adaptation of the proof of the preceding corollary one can show that the unique minimal path from any point p in the third or fourth quadrant is the part following p of any minimal path among those already discovered which passes through it. By symmetry, one obtains a unique minimal path from each point in the first two quadrants. The complete solution of the problem for the present case may therefore be stated as follows:

THEOREM 5. Let C denote the curve composed of the pieces E_n given by

$$X_n(\sigma) = \rho^n(2 - e^{b\sigma}\cos \alpha\sigma) + \Sigma_1^{n-1} \rho^k + 1$$

$$Y_n(\sigma) = -\rho^n e^{b\sigma}\sin \alpha\sigma \quad ,$$

($0 < \sigma \leq \pi/\alpha$; $\rho = e^{b\pi/\alpha}$; and $n = 0, 1, 2, \ldots$) and of the pieces obtained from these by reflection through the origin. The curve C is homeomorphic to a straight line and divides the plane into an upper and a lower part. If the right half of C is counted with the lower, the left with the upper part, the unique minimal path from a point p in the upper (resp. lower) part is obtained by following the N- (resp. P-) curve from p until it reaches C, then switching to the P- (resp. N-) curve at the point where this occurs, then following this curve until it returns to C, then switching again, and so on, until the origin is attained. In terms of the function φ, this function should be defined to be -1 at all points above or on the right half of the curve C, $+1$ elsewhere.

It should be remembered that this is stated in terms of the transformed coordinates X and Y. The result is also valid for the original coordinates x and y if C is defined as follows: Let E_0 be the

P-arc connecting the origin with the point $(\xi_1, 0)$, E_n the arc obtained by magnifying E_0 by the factor ρ^n and translating the result $|\xi_n|$ units to the right $(n = 1, 2, \ldots)$; then C is the union all the arcs E_n $(n = 0, 1, \ldots)$ together with their reflections through the origin.

§6. THE CASE $-1 < b < 0$

The preceding section contains the solution of the fundamental problem for the system (3) where $|b| < 1$, $b \geq 0$. The first of these conditions was extensively used, but the second was first used at a very late stage, namely in the first sentence following the proof of the corollary to Theorem 4. Thus all of the results listed in Section 5 down to that point are equally valid when $-1 < b < 0$.

But when $b < 0$, $\rho = e^{b\pi/\alpha} < 1$ and so $|\xi_n|$ approaches the finite limit $a = (1 + \rho)/(1 - \rho)$ as $n \longrightarrow \infty$, and Theorem 4 supplies us with minimal paths only from points on the finite interval $0 < X < a$ of the X-axis. These minimal paths sweep out a certain convex region S, and within this region, as in Section 5, we obtain a unique minimal path from each point.

This is indeed the best that could be expected; for if p lies on the boundary of S or beyond, there is no path whatever from p. This assertion is easily proved. Let $\bar{p} = (a, 0)$ and $-\bar{p} = (-a, 0)$. The boundary of S is made up of the P-arc A originating at $\bar{p}$ and terminating at $-\bar{p}$, and the N-arc B originating at $-\bar{p}$ and terminating at $\bar{p}$, as shown in Figure 8. It will suffice to prove that no N- or P-curve can move into S from any point on the boundary of S. Consider B; it is an N-arc, so N-arcs cannot move inward from it. On the other hand,

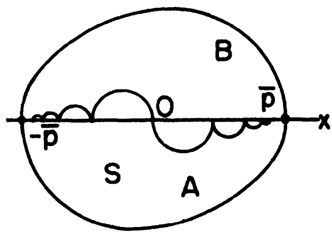

FIGURE 8

all P-curves crossing B do so moving outwards; for by the fundamental differential equations (5), an outward normal vector to B at any point (X, Y) is given by $(\alpha X + bY + \alpha, -bX + \alpha Y - b)$, while a tangent vector to the P-curve at the same point is given by $(-bX + \alpha Y + b, -\alpha X - bY + \alpha)$. The inner product of these two vectors is $2Y$, so that at all points on B strictly above the X-axis the angle between them is acute. At the end points the angle is 0, but there the radii of curvature are such that the conclusion holds good. A similar argument may be applied to A.

In the present case the solution of the problem should therefore read:

THEOREM 6. When $b < 0$, paths exist from a point p if and only if p is an interior point of S; from any such point the unique minimal path is obtained as in Theorem 5, C being defined as before.

§7. THE LINEAR CASE WITH REAL CHARACTERISTIC ROOTS

Sections 5 and 6 contain the complete solution of the problem of Section 1 for those cases in which the characteristic roots of the system

$$\dot{x} = y, \quad \dot{y} = \varphi(x, y) - (Ax + By) \tag{20}$$

are complex. When the characteristic roots are real, the matter is much simpler, and the general approach used above yields the solution much more rapidly. Since the problem has been solved in a more general form for cases of this kind (see references [6] and [1]), I shall merely state the results here and omit all proofs.

When at least one of the characteristic roots is 0, then $A = 0$ and the characteristic roots are in fact 0 and $-B$. The P-curves are obtainable from any one of them by translation along the x-axis, and the N-curves may be obtained from the P-curves by reflection through the origin. Let R denote the whole plane when $B \geq 0$, the horizontal strip $|y| < -B^{-1}$ when $B < 0$.

THEOREM 7. Paths from p exist if and only if $p \in R$; if C denotes the curve made up of the P- and N-semicurves terminating at the origin, the unique minimal path from any point p in R above (resp. below) C is obtained by following the N- (resp. P-) curve from p to C and then following C into the origin. If p is on C the unique minimal path from p is the part of C between p and the origin.

The proof of this theorem does not depend on knowing the explicit equations for the P- and N-curves, and may therefore be generalized to cover certain nonlinear cases.

When the characteristic roots for (20) are real and of the same sign, a change of scale brings (20) into the form

$$\dot{x} = y, \quad \dot{y} = \varphi(x, y) - (x + 2by) \quad ,$$

where $|b| \geq 1$. The point $(1, 0)$ is a <u>node</u> for the P-system, stable or

unstable according as b is positive or negative. Now let R denote the whole plane if $b \geq 1$, the open region bounded by the P- and N-semicurves joining $(-1, 0)$ and $(1, 0)$ when $b \leq -1$. With R redefined in this manner, Theorem 7 applies verbatim to this case.

Finally, when the characteristic roots for (20) are real and have opposite signs, the system (20) can be reduced by a suitable change of scale to the form

$$\dot{x} = y, \quad \dot{y} = \varphi(x, y) + x + 2by \quad ,$$

where b may be any real number. In this case the point $(-1, 0)$ is a saddle point for the P-system, and the N-system is, as usual, obtained by reflecting the P-system through the origin. If R is redefined as the open slanting strip bounded by the straight lines

$$y = \left(b - \sqrt{b^2 + 1}\right)\left(x \pm 1\right) \quad ,$$

Theorem 7 again applies verbatim. In this case it never happens that paths exist from every point in the plane.

§8. CONCLUSION

The preceding three sections contain the complete solution of the problem stated in Section 1 when $g(x, y)$ is assumed to be linear. It should be observed that in every case there exists a unique minimal path from each point from which there exists any path at all: the region R for which the problem is solved is the largest region for which it could possibly be solved. When the P- and N-systems are stable the region R is the whole plane; when unstable, a restricted, though not necessarily bounded, part of the plane. These observations suggest some possible general existence theorems about which nothing appears to be known.

Although, as I have pointed out, the solution for the case of real characteristic roots has been raised to a much higher degree of generality, the solution given in Section 5 for the case of complex roots has not been advanced in any direction. In view of the highly special nature of the methods used on this case, it seems that a radically different approach will be needed to obtain any generalizations.

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This paper is based on my Princeton dissertation, which was written under the guidance of Professor Lefschetz and with support from the

Office of Naval Research under contract Nonr-263(02). The dissertation was circulated in the form of [2]. The material in items [3] and [5] was available to me at that time; items [1], [4], [6], and [7] represent subsequent developments.

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Dedicated to Odon Godart

IV. ON THE STRUCTURE OF SYMMETRIC PERIODIC SOLUTIONS OF CONSERVATIVE SYSTEMS, WITH APPLICATIONS[1]

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INTRODUCTION

If one wants to study a given algebraic curve, one will start to investigate the properties of such a curve near an ordinary point, but this will not furnish any information on the behavior of the curve in the large; the knowledge of the singular points will furnish one of the tools for such a study in the large and the classification of these singular points into double points, triple points, cusps, etc., will be of utmost importance. A similar situation arises in the study of differential equations. Let us take those of the type

$$\frac{d^2x}{dt^2} = \frac{\partial}{\partial x} U(x, y), \qquad \frac{d^2y}{dt^2} = \frac{\partial}{\partial y} U(x, y) \quad ;$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 2U(x, y) \quad .$$

Here the properties of existence and uniqueness of a solution or of the behavior near a solution, do not give any information on the set of all solutions in the large; moreover it is difficult to arrive at the complete knowledge of one solution for an infinite time except for periodic solutions, and for asymptotic solutions to periodic solutions. Periodic solutions may be taken as analogous to the singular points of an algebraic curve and their knowledge and classification may furnish a good tool to advance in the study of differential equations.

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[1] This paper is extracted from a Report bearing the same title prepared under the sponsorship of the United States Air Force through the Office of Scientific Research of the Air Research and Development Command (Report No. 2 (1954), ix + 74 pp., 3 Tables, 18 Fig.).

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A classification of periodic solutions is considered here and is made dependent on a correspondence of solutions of the differential equation with points in the two dimensional space R^2. Such a correspondence can easily be established in general, if there exists a surface which is crossed by every trajectory. This surface is called surface of section by Poincaré; to successive intersections of the surface of section by the trajectories correspond successive points A and B in R^2. The correspondence of B to A defines a transformation T of R^2 into itself, and the problem of finding periodic solutions of the differential equations is equivalent to the determination of fixed points of the transformation T.

Unfortunately it is not, in general, easy to obtain a simple surface of section. Hence, it makes sense to consider the similar transformation that can be deduced from the successive intersections of part of a surface satisfying less stringent conditions. This transformation furnishes those periodic solutions which cross this surface and this gives at least a partial answer to the problem.

In a certain number of applications, such as the billiard ball problem and the restricted three bodies problem, the transformation T is the product of two involutions $M_1 M_0$. We shall restrict ourselves in this paper to such a transformation and shall study first the properties of T. Many of these properties are explicit or implicit in Birkhoff's work. It will appear that some periodic points under powers of T are contained in the intersections $\mathcal{M}_{n,p}$ of the sets $\mathcal{M}_0, \mathcal{M}_1, \ldots, \mathcal{M}_n, \ldots, \mathcal{M}_p, \ldots$ of the points invariant under M_0, M_1 and sets generalizing M_0 and M_1.

This permits a classification of the so-called symmetric periodic points $\mathcal{M}_{n,p}$.

One typical application is given in the second part of the paper and concerns symmetric periodic solutions of conservative systems of two degrees of freedom having a line of symmetry and reversible, i.e., for which any trajectory may be followed in either sense. It is shown that under some conditions the transformation T in this application is topological and the sets $\mathcal{M}_n$ are continuous curves. This imposes, when the invariant curves $\mathcal{M}_n$ are known for $n < p$, restrictive conditions on $\mathcal{M}_p$, hence the fact that some knowledge of the more complicated periodic solutions can be obtained from the less complicated ones. The conservative property of T is not explicitly mentioned here but is implicit because of the conservative system which is at the origin of T.

When a family of differential equations is considered, the periodic points corresponding to the periodic solutions vary continuously and give rise to the structure of periodic solutions. This then is investigated to a great extent for a typical conservative problem, the problem of

Störmer, bringing together all the known results, throwing light on the role of the essential singularity of the problem and presenting new results which had escaped former analysis.

As a help to the reader, we mention a more elementary application on conservative transformations in the plane which are products of involutions, which we discovered recently and of which the following mapping is an example.

$$X = -y + k(kx - y)^2 + \left[x - ky + (k^2 - 1)(kx - y)^2\right]^2$$
$$Y = x - (kx - y)^2 - k\left[x - ky + (k^2 - 1)(kx - y)^2\right]^2 \quad .$$

CHAPTER I. PERIODIC POINTS OF SYMMETRIC TRANSFORMATIONS

1. Definitions. Let us consider a one to one mapping T of a set E onto itself and an involution R of E. The study of both transformations is of interest only if some relation exists between them. We shall here consider that TR is also an involution and call T a symmetric transformation with respect to R. Such mappings, products of two involutions were already considered by Birkhoff [1, I p. 727, II p. 412, 489, 668, 718] [2, p. 186].

For the sake of simplicity we let (n being any integer)

$$P_n = T^n, \quad M_n = T^nR \quad .$$

By hypothesis, $M_0^2 = 1$, $M_1^2 = 1$ and it is easy to prove that

(1) $$T^nR = RT^{-n}$$

(2) $$P_nP_q = P_{n+q}$$

(2') $$P_nM_q = M_{n+q}$$

(2'') $$M_nP_q = M_{n-q}$$

(2''') $$M_nM_q = P_{n-q} \quad ;$$

the last relation gives, when $n = q$, the generalization of the hypothesis

$$M_n^2 = 1 \quad .$$

It follows also, that every product of a finite number of mappings T and R is a mapping P_n or M_n.

<u>2. Invariant Sets</u>. We define the subsets $\mathcal{P}_n$ and $\mathcal{M}_n$ of E as the sets of points left invariant under the mappings P_n and M_n; $\mathcal{P}_n$ for $n > 0$ is evidently the set of all the points of period n or divisor of n and $\mathcal{P}_{-n} = \mathcal{P}_n$.

We shall also consider the sets

$$\mathcal{P}_{n,q} = \mathcal{P}_n \cap \mathcal{P}_q \tag{3}$$

$$\mathcal{M}_{n,q} = \mathcal{M}_n \cap \mathcal{M}_q \tag{4}$$

where $n \neq q$; we will in general assume $n > q$.

We shall now deduce some relations; first

$$\mathcal{M}_{n,q} \subset \mathcal{P}_{n-q} \quad . \tag{5}$$

For, if $A \in \mathcal{M}_{n,q}$,

$$M_nA = A, \quad M_qA = A$$

and

$$M_nM_qA = P_{n-q}A = A \quad .$$

From a similar argument we obtain

$$\left(\mathcal{M}_q \cap \mathcal{P}_{n-q}\right) \subset \mathcal{M}_n \quad . \tag{6}$$

Also, as a corollary of (6),

$$\left(\mathcal{M}_q \cap \mathcal{P}_{n-q}\right) \subset \mathcal{M}_{n,q}$$

and because of (4) and (5),

$$\mathcal{M}_{n,q} \subset \mathcal{M}_q \text{ and } \mathcal{P}_{n-q}$$

and so,

$$\mathcal{M}_q \cap \mathcal{P}_{n-q} = \mathcal{M}_{n,q} \quad . \tag{7}$$

We remind ourselves also that

$$\mathcal{P}_n \subset \mathcal{P}_{kn} \tag{8}$$

(9) $$\mathcal{P}_{n,q} = \mathcal{P}_{(n,q)} \quad .$$

Also, if $k - k' + 1 \neq 0$

(10) $$\mathcal{M}_{n,q} \subset \mathcal{M}_{k(n-q)+n,\ k'(n-q)+q}$$

for, because of (5) and (8), if $A \in \mathcal{M}_{n,q}$,

$$P_{n-q}A = A, \quad P_{k(n-q)}A = A$$

and because of (2') and $M_nA = A$,

$$M_{k(n-q)+n}A = A \quad ;$$

similarly we have

$$M_{k'(n-q)+q}A = A \quad ;$$

the two indices in the second set of (10) must be different, hence the restriction on k and k'.

In the special case $q = 0$, $k = -1$ and $k' = -1$,

$$\mathcal{M}_{n,o} \subset \mathcal{M}_{o,-n}$$

and because of the symmetry of this relation

(11) $$\mathcal{M}_{n,o} = \mathcal{M}_{o,-n} \quad .$$

Finally corresponding to (8) and (9) we have

(12) $$\mathcal{M}_{n,o} \subset \mathcal{M}_{kn,o}$$

(12') $$\mathcal{M}_{n,1} \subset \mathcal{M}_{k(n-1)+1,1}$$

(13) $$\mathcal{M}_{n,o} \cap \mathcal{M}_{q,o} = \mathcal{M}_{(n,q),o}$$

(13') $$\mathcal{M}_{n,1} \cap \mathcal{M}_{q,1} = \mathcal{M}_{(n-1,q-1)+1,\ 1} \quad .$$

For the applications we have in mind, we introduce the following terminology: $\mathcal{M}_n$ will be called a set of symmetric points and $\mathcal{M}_{n,q}$, a set of doubly symmetric points, so that the relations (5) and (7) are a more precise form of the following theorems:

THEOREM 1. Every doubly symmetric point is periodic under T.

THEOREM 2. Every symmetric periodic point is doubly symmetric.

3. Transformations of the Invariant Sets. All the sets of symmetric points may be deduced from $\mathcal{M}_0$ and $\mathcal{M}_1$ by means of powers of the transformation T because

(14) $$\mathcal{M}_{2n} = P_n \mathcal{M}_0$$

(14') $$\mathcal{M}_{2n+1} = P_n \mathcal{M}_1$$

for, in general

(15) $$P_q \mathcal{M}_n = \mathcal{M}_{n+2q} \quad .$$

As corollary,

(16) $$P_r \mathcal{M}_{n,q} = \mathcal{M}_{n+2r,q+2r} \quad .$$

Also because of

$$\mathcal{M}_{-n} = P_{-n} \mathcal{M}_n = M_0 M_n \mathcal{M}_n = M_0 \mathcal{M}_n \quad ,$$

we have

(17) $$\mathcal{M}_{-n} = R \mathcal{M}_n \quad .$$

Up to a transformation P_n all doubly symmetric sets $\mathcal{M}_{n,q}$ may be reduced to the sets

$\mathcal{M}_{2k,o}$ when n and q are even ,

$\mathcal{M}_{2k-1,\acute{o}}$ when n is odd and q is even ,

$\mathcal{M}_{2k+1,1}$ when n and q are odd ,

$\mathcal{M}_{2k-2,-1}$ when n is even and q is odd ;

for instance, because of (16),

$$P_{-j} \mathcal{M}_{2k+2j,2j} = \mathcal{M}_{2k,o} \quad .$$

The last set may be reduced to the second one because of (16) and (11);

(18) $$P_{-k+1}\mathcal{M}_{2k-2,-1} = \mathcal{M}_{0,-2k+1} = \mathcal{M}_{2k-1,0} \quad ;$$

but no further reduction is possible.

4. Classification of Symmetric Periodic Points. Because of the Theorems 1 and 2, symmetric periodic points are doubly symmetric and conversely, so that we have to classify in a unique manner the points of the sets $\mathcal{M}_{n,q}$. But one of the iterates of these points is contained in

($\mathcal{A}$) $$\mathcal{M}_{2k,0}, \quad \mathcal{M}_{2k-1,0}, \quad \mathcal{M}_{2k+1,1} \quad .$$

and conversely, iterates of points of these sets are points in $\mathcal{M}_{n,q}$. It is thus sufficient to classify points of ($\mathcal{A}$).

Points are in different sets ($\mathcal{A}$) for different values of the indices, but because of (12) to (13') each point is contained in only one of the sets

($\mathcal{B}$) $$\mathcal{E}^{o}_{k} \qquad \mathcal{D}^{o}_{k} \qquad \mathcal{C}^{o}_{k} \qquad (k > 0)$$

defined by

$$\mathcal{E}^{o}_{k} = \mathcal{M}_{2k,0} - \underset{0<n<2k}{U}\mathcal{M}_{n,0}$$

$$\mathcal{D}^{o}_{k} = \mathcal{M}_{2k-1,0} - \underset{0<n<2k-1}{U}\mathcal{M}_{n,0}$$

$$\mathcal{C}^{o}_{k} = \mathcal{M}_{2k+1,1} - \underset{1<n<2k+1}{U}\mathcal{M}_{n,1} \quad .$$

The properties of the points of the set ($\mathcal{B}$) are as follows:

i) the period is exactly and respectively

$$2k, \qquad 2k-1, \qquad 2k$$

ii) -a point in a set $\mathcal{E}^{o}_{k}$ is in $\mathcal{M}_0$ as well as its k^{th} iterate because if $A \in \mathcal{E}^{o}_{k}$ and $P_kA = B$, then $A \in \mathcal{M}_{0,-2k}$, hence $M_0A = A$, $M_{-2k}A = A$, $M_0B = B$; similarly

-a point in the set $\mathcal{C}_k^o$ is in $\mathcal{M}_1$ as well as its k^{th} iterate and

-a point in the set $\mathcal{D}_k^o$ is in $\mathcal{M}_o$ and its $(k-1)^{th}$ iterate is in $\mathcal{M}_{-1}$.

As for the sets $(\mathcal{V})$, we have

iii) The period of the sets $\mathcal{C}_k^o$ and $\mathcal{E}_k^o$ is k and that of $\mathcal{D}_k^o$ is $2k - 1$.

The iterates of the sets $(\mathcal{V})$ will be denoted

$$\mathcal{C}_k^1, \mathcal{C}_k^2, \ldots, \mathcal{C}_k^{k-1} \quad ;$$

$$\mathcal{D}_k^1, \mathcal{D}_k^2, \ldots, \mathcal{D}^{k-1}, \mathcal{D}_k^{-k+1}, \ldots, \mathcal{D}_k^{-1} \quad ;$$

$$\mathcal{E}_k^1, \mathcal{E}_k^2, \ldots, \mathcal{E}_k^{k-1} \quad .$$

And we have

$$\mathcal{C}_k^j = R\,\mathcal{C}_k^{k-j-1} \tag{19}$$

$$\mathcal{D}_k^j = R\,\mathcal{D}_k^{-j} \tag{19'}$$

$$\mathcal{E}_k^j = R\,\mathcal{E}_k^{k-j} \quad . \tag{19''}$$

5. Other Properties. From a similar theorem on the indices follows:

THEOREM 3. The sets $\mathcal{M}_{n,o}$ are partitioned by the sets $\mathcal{E}_i$, $\mathcal{D}_j$ where $2i$ and $2j - 1$ are all the divisors of n. The sets $\mathcal{M}_{2n+1,1}$ are partitioned by the sets $\mathcal{C}_i$, $\mathcal{D}_j$ where $2i$ and $2j - 1$ are all the divisors of $2n$.

From this theorem, one may deduce the partition of any set $\mathcal{M}_{n,q}$; Table I gives the partition of $\mathcal{M}_{n,q} = \mathcal{M}_{q,n}$ corresponding to some values of n and q: $|n| \leq 4$, $|q| \leq 4$, $n \neq q$.

If the sets $\mathcal{M}_{n,p}$ are obtained in the order of increasing n and for a fixed n in the order of decreasing p, parenthesis indicate the first time the sets $\mathcal{C}_k^j$, $\mathcal{D}_k^j$, $\mathcal{E}_k^j$ are obtained for a fixed k, whatever be j; brackets indicate the first time the sets are obtained for

TABLE I

The Partitions of the Sets $\mathcal{M}_{n,q}$ are Given Below n, q.
(For the conventions see Section 5.)

$-4,-3$	$-4,-2$	$-4,-1$	$-4,0$
$\mathcal{D}_1^0$	$\mathcal{D}_1^0\ \mathcal{E}_1^0$	$\mathcal{D}_1^0\ \mathcal{D}_2^1$	$\mathcal{D}_1^0\ \mathcal{E}_1^0\ \mathcal{E}_2^0$
$-3,-2$	$-3,-1$	$-3,0$	$-4,1$
$\mathcal{D}_1^0$	$\mathcal{C}_1^0\ \mathcal{D}_1^0$	$\mathcal{D}_1^0\ \mathcal{D}_2^0$	$\mathcal{D}_1^0\ \{\mathcal{D}_3^{-2}\}$
$-2,-1$	$-2,0$	$-3,1$	$-4,2$
$\mathcal{D}_1^0$	$\mathcal{D}_1^0\ \mathcal{E}_1^0$	$\mathcal{C}_1^0\ \{\mathcal{C}_2^0\}\ \mathcal{D}_1^0$	$\mathcal{D}_1^0\ \mathcal{D}_2^1\ \mathcal{E}_1^0\ \{\mathcal{E}_3^1\}$
$-1,0$	$-2,1$	$-3,2$	$-4,3$
$\mathcal{D}_1^0$	$\mathcal{D}_1^0\ \{\mathcal{D}_2^{-1}\}$	$\mathcal{D}_1^0\ \{\mathcal{D}_3^1\}$	$\mathcal{D}_1^0\ \{\mathcal{D}_4^{-2}\}$
$1,-1$	$2,-2$	$3,-3$	$4,-4$
$(\mathcal{C}_1^0)\mathcal{D}_1^0$	$\mathcal{D}_1^0\ \mathcal{E}_1^0\ (\mathcal{E}_2^1)$	$\mathcal{C}_1^0\ (\mathcal{C}_3^1)\ \mathcal{D}_1^0\ \mathcal{D}_2^0$	$\mathcal{D}_1^0\ \mathcal{E}_1^0\ \mathcal{E}_2^0\ (\mathcal{E}_4^2)$
$1,0$	$2,-1$	$3,-2$	$4,-3$
$\longrightarrow (\mathcal{D}_1^0)$	$\mathcal{D}_1^0\ (\mathcal{D}_2^1)$	$\mathcal{D}_1^0\ (\mathcal{D}_3^{-1})$	$\mathcal{D}_1^0\ (\mathcal{D}_4^2)$
$2,1$	$2,0$	$3,-1$	$4,-2$
$\longrightarrow \mathcal{D}_1^0$	$\mathcal{D}_1^0\ (\mathcal{E}_1^0)$	$\mathcal{C}_1^0\ (\mathcal{C}_2^1)\ \mathcal{D}_1^0$	$\mathcal{D}_1^0\ \mathcal{D}_2^{-1}\ \mathcal{E}_1^0\ (\mathcal{E}_3^2)$
$3,2$	$3,1$	$3,0$	$4,-1$
$\longrightarrow \mathcal{D}_1^0$	$\mathcal{C}_1^0\ \mathcal{D}_1^0$	$\mathcal{D}_1^0\ [\mathcal{D}_2^0]$	$\mathcal{D}_1^0\ [\mathcal{D}_3^2]$
$4,3$	$4,2$	$4,1$	$4,0$
$\longrightarrow \mathcal{D}_1^0$	$\mathcal{D}_1^0\ \mathcal{E}_1^0$	$\mathcal{D}_1^0\ \mathcal{D}_2^{-1}$	$\mathcal{D}_1^0\ \mathcal{E}_1^0\ [\mathcal{E}_2^0]$

fixed k with the exception that sets related by (19) to (19'') are considered as equivalent; braces are used when this equivalence is not considered.

THEOREM 4. If all the $\mathcal{M}_i$ are known for $|i| < n$, from the knowledge of $\mathcal{M}_n$ and $\mathcal{M}_{-n} = R\mathcal{M}_n$, we deduce the new sets

$$\mathcal{E}_{n-1}, \quad \mathcal{E}_n, \quad \mathcal{D}_n \text{ when } n \text{ is even and}$$

$$\mathcal{C}_{n-1}, \quad \mathcal{C}_n, \quad \mathcal{D}_n \text{ when } n \text{ is odd.}$$

For instance, when $n = 2p$ the only new doubly symmetric sets are $\mathcal{M}_{2p,-2p+2}$, $\mathcal{M}_{2p,-2p+1}$ and $\mathcal{M}_{2p,-2p}$ because the others

$$\mathcal{M}_{2p,-2p+k} \qquad (3 \leqq k < 4p)$$

are equivalent up to a transformation P_k to

$$\mathcal{M}_{2p-2,-2p+k-2} \ .$$

The new sets are equivalent to

$$\mathcal{M}_{4p-2,o}, \quad \mathcal{M}_{4p-1,o} \quad \text{and} \quad \mathcal{M}_{4p,o} \ ;$$

when these sets are partitioned with the aid of Theorem 3, only the improper divisors give new sets $(\mathcal{E})$, hence the theorem.

6. Non Existence of Symmetric Periodic Points. Let us consider two sets E_1 and E_2 which form with $\mathcal{M}_o$ a partition of E, we have

THEOREM 5. If the mappings of E_1 and $\mathcal{M}_o$ are contained in E_1, the sets $\mathcal{D}_n$ and $\mathcal{E}_n$ are empty.

For, $T(E_1 \cup \mathcal{M}_o) \subset E_1$ is equivalent to

$$T(\mathcal{C}E_2) \subset E_1$$

from which we deduce

$$E_2 \supset \mathcal{C}(T^{-1}E_1) = T^{-1}(\mathcal{C}E_1)$$

and so

$$\mathcal{M}_{-2} = T^{-1}\mathcal{M}_o \subset E_2;$$

hence $\mathcal{M}_{o,-2} = \emptyset$; on the other hand by iteration of the hypothesis, $T\mathcal{M}_o \subset E_1$ and $T(E_1) \subset E_1$,

$$\mathcal{M}_{2p} = T^p\mathcal{M}_o \subset E_1$$

and so

$$\mathcal{M}_{2p,-2} = \emptyset \quad \text{for} \quad p > 0$$

but the sets $\mathcal{D}_n^o$ and $\mathcal{E}_n^o$ are contained in some $T\mathcal{M}_{2p,-2} = \emptyset$ $(p \geq o)$ and so these sets and their iterates are empty.

If E_3, E_4 and $\mathcal{M}_{-1}$ form a partition of E, it is easy to prove

THEOREM 6. If the mapping of E_3 and the mapping of $\mathcal{M}_{-1}$ are contained in E_3, the sets $\mathcal{C}_n$ and $\mathcal{D}_n$ are empty.

Also if E_3 and E_4 form a partition of E we have

THEOREM 6 bis. If $\mathcal{M}_1$ and the mapping of E_3 are contained in E_3 and if $\mathcal{M}_{-1}$ is contained in E_4, the sets $\mathcal{C}_n$ and $\mathcal{D}_n$ are empty.

CHAPTER II. APPLICATION TO A CLASS OF CONSERVATIVE PROBLEMS OF TWO DEGREES OF FREEDOM

The properties of Chapter I were suggested by the following application.

Let us consider the canonical form of a conservative dynamical problem of two degrees of freedom

$$\ddot{x} = \partial U/\partial x \tag{20}$$

$$\ddot{y} = \partial U/\partial y \tag{21}$$

for which the following first integral is easily found

$$\dot{x}^2 + \dot{y}^2 = 2U(x, y, a) + h \quad . \tag{22}$$

Let us think of U as an analytic function in x and y; weaker conditions are studied in Section 13. In the problems which are naturally of two degrees of freedom, the constant a does not appear and h is the integration constant; when the equations are deduced from a problem of three degrees of freedom with an ignorable coordinate, the constant a is the

conjugate coordinate and often h is fixed if a suitable system of units is chosen. Hence there is no restriction if we write $h = 0$ or

$$(22') \qquad \dot{x}^2 + \dot{y}^2 = 2U(x, y, a) \quad .$$

Different spaces may be used:

1) The four dimensional phase space,
2) The three dimensional surface (22') if a is fixed,
3) A two dimensional surface of section [20], [I, II p. 70], if a is fixed; but such a surface does not exist in general.
4) The two dimensional subspace $\mathcal{S}$ of the plane $x, y : 2U \geqq 0$.

7. Introduction of Symmetry. We shall now make the further hypothesis $U(x, -y, a) = U(x, y, a)$, which means that the problem is symmetrical with respect to $y = 0$.

$y = 0$ is a solution of (21) and the solution on the axis of symmetry is reduced to a problem of one degree of freedom (20).

When a is fixed, a solution of the problem which crosses $y = 0$, is determined up to a symmetry by the values of x and $\dot{x}$ at the time of crossing because $\dot{y}$ is determined up to the sign by (22'). The two dimensional subspace $\mathcal{T}$ of the plane $x, \dot{x}$

$$2U(x, 0, a) - \dot{x}^2 \geqq 0 \quad ,$$

is a generalization of a surface of section: this surface is made up of analytic pieces, but some discussion will have to be made when there are double points on the boundary and for regular boundedness [I, II, p. 70]; we do not ask that all the trajectories cross $y = 0$.

If we are looking for periodic solutions of (20) and (21), symmetry is very important, because then symmetric periodic solutions may exist and these are much more easy to find than non-symmetric periodic orbits as will be indicated in Section 10. We shall consider here the symmetry defined above, but what follows may be generalized to other cases.

The two subspaces $\mathcal{S}$ and $\mathcal{T}$ will mostly be used. Each point A of $\mathcal{T}$ not on the boundary gives the initial conditions of two symmetric curves τ_A in $\mathcal{S}$ crossing the line of symmetry $y = 0$ at $\bar{A}$ and followed in a definite sense and conversely. The points on the boundary of $\mathcal{T}$ correspond to the solution $y = 0$, $\dot{x}^2 = 2U$.

8. Transformations of the Set $\mathcal{T}$. Let us consider a trajectory corresponding to a point A of $\mathcal{T}$; if this trajectory again crosses $y = 0$, then to this point A there corresponds a point B of $\mathcal{T}$ and the differential equations define a transformation $\overline{T}$ of $\mathcal{T}$ into itself, $\overline{T}^{-1}$ is obtained by following the same trajectory in the other sense giving, if the trajectory crosses $y = 0$ at D: $\overline{T}^{-1} A = D$. Let us now define the involution $\overline{R}$ of $\mathcal{T}$ as the reflexion about $\dot{x} = 0$; this corresponds in $\mathcal{S}$ to the trajectory starting at the same point on $y = 0$ in the opposite direction. If $\overline{R}A = C$, we have $\overline{TR}C = B$ and $\overline{TR}B = C$ and so $\overline{TR}$ is also an involution.

If $\overline{T}$ is a one to one mapping of $\mathcal{T}$ onto itself, i.e., if to each A corresponds a B and a D, then the application of Part I is immediate and $E = \mathcal{T}$; if not, E will be the largest subset of $\mathcal{T}$ for which the transformation T deduced from $\overline{T}$ is one to one onto E. An *a priori* knowledge of E is then, in general, difficult to obtain, but is not necessary.

9. Periodic Orbits and Symmetric Periodic Orbits. Solutions of (20) and (21) will be called here indifferently, trajectories or orbits. It is immediate that the sets $\mathcal{P}_n$ of E correspond to the periodic orbits τ crossing $y = 0$ at least once and conversely. We have now to find the meaning of the sets $\mathcal{M}_n$. $\mathcal{M}_0$ is the set of points invariant under R, i.e., the set of points on $\dot{x} = 0$ in E. These points correspond to trajectories perpendicular to $y = 0$ in $\mathcal{S}$, i.e., to trajectories symmetric with respect to the x axis or trajectories symmetric with respect to the plane $y = 0$, $\dot{x} = 0$ in the phase space. $\mathcal{M}_{2n}$ is the transform of $\mathcal{M}_0$ under T^n (14); these points corresponds to the n^{th} crossing of trajectories symmetric with respect to $y = 0$. If A is a point of $\mathcal{M}_1$, TRA = A; to A corresponds in the phase space a trajectory τ_A with initial conditions $X(x_0, y_0 = 0, \dot{x}_0, \dot{y}_0)$, to RA corresponds the trajectory τ_{RA} with initial conditions $Y(x_0, 0, -\dot{x}_0, -\dot{y}_0)$ and

$$\dot{x}(-t, X) = -\dot{x}(t, Y) \quad .$$

After a time $2t_1$, τ_{RA} crosses $y = 0$ at X, and we have, because of TRA = A

$$\dot{x}(2t_1, Y) = \dot{x}(0, X) \quad ;$$

but because of the uniqueness in the phase space, this was true at any preceding time, for instance $-t_1$, and so

$$\dot{x}(t_1, Y) = \dot{x}(-t_1, X) = -\dot{x}(t_1, Y) = 0 \quad .$$

For the same reason $\dot{y}(t_1, Y) = 0$. The orbit will have one point of zero velocity or in $\mathcal{S}$ a point on the boundary $2U = 0$, and A is the next point of intersection with $y = 0$. Because of (14') $\mathcal{M}_{2n+1}$ is the set in E of the orbits starting from the zero velocity line and crossing $y = 0$ for the n^{th} time. These orbits are symmetric in the phase space about $\dot{x} = 0$, $\dot{y} = 0$. Symmetric points correspond to symmetric orbits in the preceding sense.

The sets $\mathcal{M}_n$ are defined in E, but we extend their definition in $\mathcal{T}$ in the obvious manner; they will be, in general, lines which intersect in doubly symmetric points $\mathcal{M}_{n,q}$, which correspond (Theorem 1) to symmetric periodic orbits; the converse is also true because of Theorem 2.

10. On the Advantage of Symmetry. Actually the sets $\mathcal{M}_n$ in $\mathcal{T}$ will be obtained by integrating the differential equations (20) and (21) with initial conditions $(x, y = 0) \in S$, $\dot{x} = 0$, $\dot{y}$ given by (22') or with the initial conditions (x, y) satisfying $U(x, y) = 0$, $\dot{x} = \dot{y} = 0$. The sets $\mathcal{M}_{n,q}$ do not give all the periodic solutions, namely those which do not cross $y = 0$, and those which cross $y = 0$ and are not symmetric. The last ones may in general only be obtained with much more work, because we have then to integrate all the solutions starting with any value of x, $\dot{x}$ in $\mathcal{T}$ up to the second crossing with $y = 0$ in $\mathcal{S}$; by using the symmetry argument, the integration with the indicated initial conditions up to the first crossing with $y = 0$ is sufficient.

Periodic orbits which do not cross $y = 0$ may be obtained in the case where there exists another plane of symmetry in the phase space, which could be for instance the $x = 0$, $\dot{y} = 0$ plane.

But to deduce from this, that all non-symmetric periodic solutions are obtained, we must have more knowledge on the surfaces of section and prove that every trajectory crosses in the phase space either $x = 0$, $\dot{y} = 0$ or $y = 0$, $\dot{x} = 0$.

11. Classification of Periodic Orbits. Because of the correspondence of the symmetric periodic points and the symmetric periodic orbits, Section 4 gives us a classification of symmetric periodic orbits.

With the notation introduced at the end of Section 7, if

$$A \in \mathcal{E}_k^o \subset \mathcal{M}_{2k,o}$$

we have with $B = T^k A$,

$$M_o A = A, \quad M_o B = B \quad \text{(Section 4, ii)}$$

$$\mathcal{E}_k^i \ni A_i = T^iA = T^iRA = RT^{-i}A = R(T^{k-i}B) \qquad i = 1, 2, \ldots, k-1$$

all the points A_i being different because of the definition of $\mathcal{E}_k^o$. The corresponding trajectory τ_A crosses $y = 0$ perpendicularly at $\overline{A}$ and $\overline{B}$ and has exactly $k - 1$ other points of intersection $\overline{A}_i$ with $y = 0$, such a trajectory will be noted ϵ_k.

Similarly to $A \in \mathcal{C}_k^o$, corresponds a trajectory γ_k with exactly k points of intersection with $y = 0$ and with two points of zero velocity. Also to $A \in \mathcal{D}_k^o$, corresponds a trajectory δ_k with exactly k points of intersection with $y = 0$ at one of which $(\overline{A})$ the crossing is perpendicular and with two points of zero velocity symmetric with respect to $y = 0$.

Using Table 1, one sees that from the computation of $\mathcal{M}_1$, δ_1 and γ_1 may be obtained, i.e., the periodic orbits symmetric or not with respect to $y = 0$ and having two points of zero velocity. The additional computation of $\mathcal{M}_2$ gives every orbit ϵ_1, ϵ_2 and δ_2, i.e., the symmetric periodic orbits with respect to $y = 0$ with two points where the crossing is perpendicular and without or with an additional crossing point and the symmetric periodic orbits with respect to $y = 0$ with two symmetric points of zero velocity and two points on $y = 0$, one at which the crossing is perpendicular. And so on.

12. <u>Non Existence of Symmetric Periodic Solutions</u>. If in the Theorem 5, $E_1 = (\mathcal{J} \cap \dot{x} > 0)$ and $E_2 = (\mathcal{J} \cap \dot{x} < 0)$ one has

THEOREM 7. If all trajectories starting from $y = 0$ with $\dot{y} > 0$ and $\dot{x} \geq 0$ and which cross $y = 0$ are such that at the first crossing point $\dot{x} > 0$, there is no symmetric periodic orbit of type δ or ϵ.

Moreover, if $E_3 = E_1$ and $E_4 = E_2$ we derive from Theorem 6 bis:

THEOREM 8. If the hypothesis of Theorem 7 are verified and if all trajectories starting with zero velocity and $y > 0$ and which cross $y = 0$ are such that at the first crossing point $\dot{x} > 0$, there is no symmetric periodic orbit.

The additional hypothesis means that $\mathcal{M}_1 \subset E_3$ hence

$$\mathcal{M}_{-1} = R\mathcal{M}_1 \subset RE_3 = E_4 \quad .$$

REMARK. As such the two preceding theorems are not very useful, but will become so when T is a topological transformation.

13. Properties of the Transformation T.

THEOREM 9. Let $\mathcal{S}' \subset \mathcal{S}$ be a closed bounded domain in which $U(x, y, a)$ has continuous bounded first partial derivatives in x and y and in which these derivatives satisfy Lipschitz's condition. Let $I(x_0)$ be a closed set of intervals in x_0 such that

$$\Big(I(x_0),\ y_0 = 0\Big) \in \mathcal{S}'$$

and $\mathcal{T}'$ such that

$$\mathcal{T}' = \mathcal{T} \cap \Big(I(x_0),\ \dot{x}_0\Big) \quad ,$$

then to every point $P_0 \in \mathcal{T}'$ corresponds a unique solution of the differential equations (20) and (21); if this solution crosses again $y = 0$ after a finite time t_1, and before leaving $\mathcal{S}'$, the corresponding point P_1 is in a set $\mathcal{T}_1'$; and the original point is in the set $\mathcal{T}_0'$; the correspondence between $\mathcal{T}_0'$ and $\mathcal{T}_1'$ is one to one.

PROOF. The system of first order differential equation (20) and (21) satisfies the condition for existence and uniqueness of a solution with initial conditions at $t = t_0$, $(x_0, y_0) \in \mathcal{S}'$ and $\dot{x}_0$, $\dot{y}_0$ satisfying (22); moreover those solutions are continuous functions of x_0, y_0, $\dot{x}_0$, $\dot{y}_0$ and $(t - t_0)$ in $\mathcal{S}'$ [28, II, 141]; hence the correspondence between some subsets of $\mathcal{T}$ is one to one and it is easy to see that $\mathcal{T}_0'$ and $\mathcal{T}_1'$ are such subsets. It is not so that those solutions are necessarily continuous functions of the initial conditions alone, for, t may increase indefinitely along a solution without these solutions leaving $\mathcal{S}'$, for instance when the solution is asymptotic to a periodic solution.

THEOREM 10. If besides the hypothesis of Theorem 9, U admits in $\mathcal{S}'$ continuous bounded second partial derivatives which satisfy a Lipschitz condition, the correspondence between $\mathcal{T}_0'$ and $\mathcal{T}_1'$ is continuous and thus topological.

We shall restrict ourselves at first to the open domains of $\mathcal{T}_0'$ and $\mathcal{T}_1'$, a solution corresponding to a point in $\mathcal{T}'$ has then continuous first partial derivatives in x_0, y_0, $\dot{x}_0$, $\dot{y}_0$ and $t - t_0$. In particular, if $y_0 = 0$ and $\dot{y}_0$ is determined by (22), y is a continuous function of x_0, $\dot{x}_0$, and $t - t_0$, since $\dot{y}_0$ is by (3) a continuous function of x_0 and $\dot{x}_0$. Now y becomes zero for $t_1 = t - t_0$, is differentiable and its derivative with respect to t does not become zero at P_1 (otherwise the trajectory would be on $y = 0$ and P_1 on the boundary of $\mathcal{T}'$), nor in the vicinity of P_1 (since $\ddot{y}_1$ is bounded by (21)). Hence there exists a unique differentiable function $t_1 = \varphi(x_0, \dot{x}_0)$ which satisfies identically $y(x_0, \dot{x}_0, t_1) = 0$ [28, I, 141-142]. If we replace t_1 by φ in the continuous and differentiable functions x and $\dot{x}$, we obtain a result even stronger than the one stated.

We have now to discuss points on the boundary of $\mathcal{T}'$. If such a point is not on the boundary of $\mathcal{T}$, it is immediate that x and $\dot{x}$ are continuous and differentiable for any variation of x_0, $\dot{x}_0$ which points to the interior of $\mathcal{T}'$. If the point is on the boundary of $\mathcal{T}$, we need to consider more closely the solutions of the differential equations near $y = 0$. If $x(t)$, $y = 0$ is a solution, any solution in the vicinity $x + \xi$, η, satisfies the differential equations.

$$\ddot{\xi} = \left(\frac{\partial^2 U}{\partial x^2} \right)\xi \tag{23}$$

$$\ddot{\eta} = \left(\frac{\partial^2 U}{\partial y^2} \right)\eta \tag{24}$$

$$\dot{\zeta}\dot{x} = \ddot{x}\zeta \tag{25}$$

with errors of the type $\epsilon(|\xi| + |\eta|)$, ϵ tending to zero with ζ and η if the time is bounded.

$$\left(\frac{\partial^2 U}{\partial x \partial y} \right) = 0$$

because of the symmetry of U. The first equation is equivalent to the third, and (25) shows that the variation ξ is of the form $A\dot{x}$. For any initial condition $\eta = 0$, $\dot{\eta} \neq 0$, the solution of the second equation at any finite time cannot be zero at the same time as $\dot{\eta}$, and so at P_1, $\eta = 0$ can be solved explicitly in t and continuity and differentiability follow easily for points on the boundary of $\mathcal{T}$, for directions interior to $\mathcal{T}'$. The proof of the theorem is completed if the points on the boundary are considered as limit points of corresponding sequences in the interior of $\mathcal{T}$. This asks for the condition of finite time entering in

the definition of $\mathcal{T}'$. The following interpretation must then be given to points L on the boundary of $\mathcal{T}$: if L is the limit of a sequence of points L_1 interior to $\mathcal{T}$, the time t_1 after which the corresponding trajectories cross $y = 0$ tends to a finite time t as L_1 tends to L. Theorems 9 and 10 may serve to prove under certain conditions, continuity of the curves $\mathcal{M}_{2n}$ as iterates of $\mathcal{M}_0$. It is also possible to write similar conditions which will serve to prove continuity for $\mathcal{M}_{2n+1}$ as obtained from a set of initial conditions on $U = 0$.

To investigate other properties of the curves $\mathcal{M}_n$, many hypotheses may be studied. We shall give as illustration the following theorems and considerations.

14. The Case of a Connected Bounded Domain $\mathcal{S}$.

THEOREM 11. If the domain $\mathcal{S}$ is bounded, connected, and containing a segment PQ on $y = 0$; if in this closed domain $U(x, y, a)$ has continuous bounded second partial derivatives in x and y and these derivatives satisfy Lipschitz conditions; if PQ is a line of section, i.e., if every trajectory through any point crosses PQ after a finite positive time, the correspondence T is topological and the invariant curves $\mathcal{M}_n$ are continuous.

This follows quite directly from the two preceding theorems. In this case the set $I(x_0)$ is the closed segment PQ and $\mathcal{T}_0' \equiv \mathcal{T}_1' \equiv \mathcal{T}$. The invariant curves $\mathcal{M}_{2n}$ are the iterates of the continuous segment PQ, and the curves $\mathcal{M}_{2n+1}$ are iterates of $\mathcal{M}_1$ which is continuous by a reasoning analogous to the one in Theorems 9 and 10 because of the continuity with respect to the initial conditions corresponding to the curve $U = 0$, $y \geqq 0$.

The end points of the curves are on the boundary of $\mathcal{T}$, and correspond to trajectories infinitely close to the line of symmetry. Properties of the immediate vicinity of $y = 0$ may hence furnish some properties on symmetric periodic orbits; for instance.

THEOREM 12. Let us consider two trajectories p and q infinitely near the line of symmetry, such that $\dot{y} = 0$ (hence $\dot{x} = 0$) at P and Q respectively and limited to a section corresponding to the segment PQ; if their total number of intersections with the open segment PQ is m, under the conditions of Theorem 11, there exists an odd number of periodic solutions of type

8 if m is different from one and an even number if m equals one. The case where p or q crosses PQ at P or Q requires a special investigation. It is also necessary for the hypothesis of Theorem 11 to be fulfilled that the periodic orbit on $y = 0$ has a non-real characteristic exponent $(k < 1)$.

The solutions near the axis of symmetry are given by Hill's equation (see, for instance, [5])

$$\ddot{\eta} = \left(\frac{\partial^2 U}{\partial y^2} \right)\eta \tag{26}$$

where the partial derivative is computed on the periodic solution. The Sturm-Liouville Theorem applies and the roots of two solutions of (26) separate each other, so that if one of the sections of the trajectories considered has at least two points of intersection with PQ, the other will have at least one. However, the first point of intersection of both sections gives one end point of $\mathscr{M}_1$, hence if $m > 2$ both end points are in regions of $\mathscr{T}$ separated by $y = 0$; if $m = 2$ both sections must have one intersection with PQ and $\mathscr{M}_1$ has also an odd number of intersections with PQ; if $m = 1$ both end points are not separated by PQ and the number of intersections with PQ is even; if $m = 0$, we must use Korteweg's Theorem [29, p. 404] which states that if η_p, η_{p+1}, η_{p+2} are the values of η for $t = pT$, $(p + 1)T$, $(p + 2)T$ where T is the period of the orbit on $y = 0$, we have

$$\eta_{p+2} + \eta_p = 2k\, \eta_{p+1}, \qquad k = \text{constant} = \cosh \Omega T, \tag{27}$$

Ω being called the characteristic exponent. Hence the trajectories p, q, when extended, will be such that at P and Q or Q and P their ordinate η is proportional to

$$1,\ \eta_1,\ d_1,\ c_1,\ \eta_1 d_2,\ \dots,\ c_n,\ \eta_1 d_{n+1},\ \dots \tag{28}$$

$$1,\ \eta_2,\ d_1,\ c_1,\ \eta_2 d_2,\ \dots,\ c_n,\ \eta_2 d_{n+1},\ \dots \tag{29}$$

with $c_n = \cosh n\Omega T$

$$d_n = 2 \cosh \Omega T - d_{n-1} - d_{n-2} \quad ,$$

$$d_{-1} = d_1 = \cosh \frac{\Omega T}{2} \quad \text{or} \quad d_n = \cosh (n + 1/2)\Omega T \quad .$$

If the orbit on $y = 0$ has an odd instability, [19], [5], i.e.,

$k = c_1 \leq -1$, then the second and third members of both series (28) and (29) alternate in sign, for $\eta_1 d_1$ and $\eta_2 d_2$ are positive $(m = 0)$ and so the end points of η_1 are separated by $\dot{x} = 0$.

If $k > -1$, we can always choose Ω so that d_1 is positive, hence η_1 and η_2 are positive. If now two consecutive terms of the series (28) are of opposite sign, it will be so for the corresponding terms of (29). The first alternation in sign must occur for corresponding couples in the two series and once again the end points of $\mathcal{M}_1$ are separated by $\dot{x} = 0$.

The exceptional case corresponds to a series for which c_n and d_n are all positive, implying $k = \cosh \Omega T \geqq 1$. In the case $k = 1$, there exists at least one periodic solution for η. That p is such a solution is impossible, for if we take the family of orbits with their initial points on $U = 0$ tending to P, the time of the first crossing would become infinite, as indicated by the behavior of p; in all other cases the solution for η is of the form

$$t\phi(t) + \Psi(t)$$

where ϕ and Ψ are periodic in t with the same period as the orbit on $y = 0$. This indicates that this orbit is unstable, and the same is true if $k > 1$. In these cases the time of crossing of p or q may be made as large as we want as the initial η becomes small. This however is excluded by hypothesis. (See end of proof of Theorem 10.) In the case $k = -1$, at least one of the orbits, p or q, crosses PQ at Q or P and the variational equations of first order (26) is not sufficient to derive the behavior of $\mathcal{M}_1$ near its end point.

As a corollary we mention that the number of intersections of the curves $\mathcal{M}_1$ and $\mathcal{M}_2$ is of the same parity because every intersection with PQ of $\mathcal{M}_2$ and not $\mathcal{M}_1$ corresponds to a curve ϵ_1 which has two points on PQ. Other topological properties of the invariant curves may be deduced from Section 1; for instance, $\mathcal{M}_p$ and $\mathcal{M}_q$ $(p, q > 0)$ intersect only at known points or at symmetric points of known points. It is then possible, if all $\mathcal{M}_{n's}$ are known for $n < p$, to determine a domain, sometimes much smaller than $\mathcal{S}$ which contains $\mathcal{M}_p$, since most of its intersections with $\mathcal{M}_0$ and all of its intersections with $\mathcal{M}_n (n < p)$ are known.

This in turn will furnish an approximation of the new periodic solutions to which the knowledge of $\mathcal{M}_p$ leads.

15. Continuous Variation of the Parameter a. A continuous variation of the parameter a will now be considered. In this case, if U is a continuous function of a, the invariant curves will vary continuously,

this indicates that the periodic points appear and disappear in pairs [20] except if the point of disappearance is on the boundary of $\mathcal{T}$ (and on $\mathcal{M}_0$); in this case, an orbit disappears by flattening on $y = 0$, this may be given the interpretation that two symmetric orbits disappear on $y = 0$, they coincide if they are of the type δ or ϵ. If two points in $\mathcal{E}_k$ appear, they give only one orbit ϵ_k and an interpretation must be given to the theorem of Poincaré referred to.

At first sight one would think that in general the contact of $\mathcal{M}_k$ and $\mathcal{M}_0$ will be of first order but just as often the contact may be of second order. To make this clear let us consider for instance a family of periodic solutions of type δ_1, the invariant $k = \cosh \Omega\overline{T}$ (1) is a continuous function of the parameter and may take values $+1$, or -1, in this case orbits near δ_1 appear or disappear. This has been studied in [9]. The results are as follows: When $k = -1$, either

a) $\mathcal{M}_1$ is orthogonal to $\mathcal{M}_0$ and near δ_1 exist orbits of type γ_1, or

b) $\mathcal{M}_2$ has an inflection point and near δ_1 exist orbits of type ϵ_1. Inflection point in this context means that the order of contact is at least two.

When $k = 1$, either

c) there exist near δ_1 non symmetric periodic orbits which cross δ_1 near the points of zero velocity and have a double point on $y = 0$, or

d) $\mathcal{M}_1$ and $\mathcal{M}_2$ have an inflection point and near δ_1 exist orbits of type δ_1 distinct from the original family.

Conversely, if at a point where $\mathcal{M}_1$ crosses $\mathcal{M}_0$

i) $\mathcal{M}_1$ is orthogonal to $\mathcal{M}_0$, we have $k = -1$, case a) and orbits γ_1,

ii) $\mathcal{M}_1$ has an inflection point with $\mathcal{M}_0$, it will be so for $\mathcal{M}_2$ and $k = +1$, case d) and orbits δ_1,

iii) $\mathcal{M}_2$ has an inflection point with $\mathcal{M}_0$, but not $\mathcal{M}_1$, we have $k = -1$, case b) and orbits ϵ_1,

iv) $\mathcal{M}_2$ is orthogonal to $\mathcal{M}_0$, $k \neq \pm 1$, there exists an orbit in the vicinity of δ_1 which starts orthogonally to $y = 0$, crosses δ_1 after $t = \overline{T}$

(1) $\overline{T}$ is here the half period $T/2$ i.e., the time between two consecutive crossings of $y = 0$ by δ_1, this is the natural definition as follows from [9].

on $y = 0$ and will be again orthogonal to $y = 0$ after $t = 2\overline{T}$, this means that $\cosh \Omega T = -1$ or $\cosh \Omega\overline{T} = 0$ and that the orbit is of type ϵ_2.

v) $\mathcal{M}_2$ is tangent to $\mathcal{M}_{-1}$ but no other preceding property is satisfied; $k = \pm 1$, there exists orbits in the vicinity of δ_1 which starts with zero velocity crosses $y = 0$ the second time orthogonally; in this case after $t = 3\overline{T}$ again the velocity is zero and $\cosh \Omega 3\overline{T} = 1$, hence $\cosh \Omega\overline{T} = -\frac{1}{2}$ and the orbits are of type δ_2. The points of zero velocity being on one side of δ_1, we may expect for values of γ_1 nearby two orbits of type δ_2, one with the points of zero velocity on one side of δ_1, and one with points of zero velocity on the other side. Case c) can not be discovered from the behavior of $\mathcal{M}_1$ and $\mathcal{M}_2$.

The similar results for periodic solutions of type ϵ_1 crossing $y = 0$ at X and Y as follows: When $k = -1$,

a'), b') $\mathcal{M}_2$ is orthogonal to $\mathcal{M}_0$ either at X or at $Y = TX$ and near ϵ_1 exist orbits of type ϵ_2.

When $k = +1$, either

c') there exists near ϵ_1 a non symmetric orbit crossing ϵ_1 at X and T, or

d') $\mathcal{M}_2$ has an inflection point with $\mathcal{M}_0$ and near ϵ_1 exist orbits of the same type ϵ_1 distinct from the original family. Conversely, if at a point where $\mathcal{M}_2$ crosses $\mathcal{M}_0$ but not $\mathcal{M}_1$

vi) $\mathcal{M}_2$ is orthogonal to $\mathcal{M}_0$, we have $k = -1$, case a) or b) and orbits ϵ_2,

vii) $\mathcal{M}_2$ has an inflection point with $\mathcal{M}_0$, we have $k = +1$, case d) and orbits ϵ_1; case c') can not be discovered from the behavior of $\mathcal{M}_2$.

Another interesting property is that if for a certain value of $a = a_0$ one orbit of type ϵ_1 appears, corresponding to I on $\dot{x} = 0$, $\mathcal{M}_2$ must have a contact or order at least two with $\dot{x} = 0$ at I. Let us note the order of the contact at I by k_1 for $\mathcal{M}_1$ and k_2 for $\mathcal{M}_2$, with the convention $k = 0$ when the curve $\mathcal{M}$ crosses $\dot{x} = 0$ at I but is not tangent at $\dot{x} = 0$ and $k = -1$ when $\mathcal{M}$ does not pass through I.

We know that $k_2 \geq 1$, we must disprove $k_2 = 1$. If k_2 would be one, $\mathcal{M}_2$ would cross $\dot{x} = 0$ for a let us say smaller than a_0 at two points I_1 and I_2, these points are of period two under the transformation T. If $I_2 = T(I_1)$, and a increases to a_0, I_2 and I_1 tend to I and the orbit at I is of type δ_1 and not ϵ_1 unless $k_2 > 1$. If $J_1 = T(I_1)$ and $J_2 = T(I_2)$ when a increases to a_0, I_3 and I_4 tend to a point J iterate of I, distinct from I, this means that for $a = a_0$ two orbits of type ϵ_1 appeared and not one, this is not an ordinary situation.

CHAPTER III. APPLICATION TO STÖRMER'S PROBLEM

15. Generalities. As application of the preceding results, we will choose a typical conservative problem of two degrees of freedom which is of interest in physics, namely the study of the motion of an electrical particle in the magnetic field of an elementary dipole; the Lagrangian of the problem is

$$L = T + e\,\vec{v} \cdot \vec{A}$$

where T is the kinetic energy, e the charge of the particle, $\vec{v}$ its velocity and $\vec{A}$ the potential vector of a magnetic dipole of moment M situated at the origin in the direction of the z axis; this vector in polar coordinates may be taken as tangent to a parallel, positive in the West direction and of length $M \cos \lambda / r^2$; this gives the Hamiltonian

$$H = \frac{1}{2m}\left[p_r^2 + r^{-2}p_\lambda^2 + \left((r^{-2}\cos^{-2}\lambda)p_\varphi - Mer^{-1}\cos\lambda\right)^2\right]$$

hence the Hamiltonian and the momentum p_φ are constant [27]. When p_φ is not positive, Störmer has proven [21, p. 23] that no periodic solution exists; when p_φ is positive, with as unities m, v and Me and with the change of variable

$$p_\varphi r = e^x, \quad d\sigma = e^{-2x}p_\varphi^3\,dt$$

the equations deduced from the new Hamiltonians are:

$$\ddot{x} = ae^{2x} - e^{-x} + e^{-2x}\cos^2\lambda \equiv X = \partial U/\partial x \tag{30}$$

$$\ddot{\lambda} = \left(e^{-2x}\cos^2\lambda + 1 + \tan^2\lambda\right)\tan\lambda \equiv \Lambda = \partial U/\partial\lambda \tag{31}$$

$$\dot{x}^2 + \dot{\lambda}^2 = ae^{2x} - 1 - \tan^2\lambda + 2e^{-x} - e^{-2x}\cos^2\lambda \equiv 2U \tag{32}$$

here $a = p_\varphi^{-4}$ is used, $\gamma_1 = -\gamma = p_\varphi/2$ are equivalent parameters. To every solution in the meridian plane r, λ correspond an infinity of trajectory in the space obtained by integrating the equation for the longitude:

$$\dot{\varphi} = \cos^{-2}\lambda - e^{-x} \quad . \tag{33}$$

This integration is a trivial problem and the properties of the trajectories in the three dimensional space are deduced easily from those in the meridian plane, for instance to a periodic solution of period Σ in this plane correspond trajectories on the surface of revolution around the z axis having the periodic solution as directrix. If

$$\Phi = \int_0^\Sigma \left(\cos^{-2}\lambda - e^{-x}\right) d\sigma$$

is commensurable with π, all corresponding trajectories are periodic in space, if Φ is incommensurable each trajectory recurs infinitely often to the neighborhood of an initial state on the surface of revolution and may be classified as almost periodic; hence the general problem is easily solved if the problem in the x, λ plane, is; for an example see [11].

Let us now consider some general properties of the equations (30, 31).

16. <u>Properties</u>. The equations are of the type discussed in Part II, λ playing the role of the variable y, the x axis is the axis of symmetry; the portion $\mathcal{S}$ of the x, λ plane, $U \geqq 0$, has been described as follows (see for instance [27]), the boundary $U = 0$ is made up of a branch asymptotic for $x = -\infty$ to $\lambda = \pi/2$ cutting the x axis orthogonally at P with $x = g$ and when $0 < a < 1/16$ of two similar branches one asymptotic for $x = -\infty$ to $\lambda = \pi/2$ cutting $\lambda = 0$ at Q with $g_1 > g$, the other asymptotic for $x = +\infty$ to $\lambda = \pi/2$ cutting $\lambda = 0$ at $g_2 > g$; $\mathcal{S}$ is then formed of two disconnected regions; when $a > 1/16$ the two last branches become connected and do not cross $\lambda = 0$, $\mathcal{S}$ is then formed of one connected region. When $a = 1/16$ the boundary has a double point at $x = \log 2$, $\lambda = 0$ and a special discussion is needed. When $a = 0$, $\mathcal{S}$ reduces to a line

$$e^x = \cos^2\lambda \tag{34}$$

called the thalweg, all points of this line are equilibrium points and this case will be excluded in the following.

For every finite point X and Λ are analytic; by a change of variable one finds that the points at infinity are regular with one

exception. The point $x = -\infty$, $\lambda = \pi/2$ is an essential singularity and has been studied by Störmer [22, p. 145, 235], [23], [26, 1947, 1949], the proof that there exists a solution through that point has been furnished by Malmquist [18], but no proof of uniqueness is available although this seems highly probable.

The general results on periodic solutions are as follows: every periodic solution is in the domain $x < f = \log (2\gamma_1^{4/3}) < g_2$ [25, p. 61]; this shows that no periodic solution extends to $+\infty$ and exists in the region which extends to $+\infty$ in the case $a < 1/16$. Every periodic solution crosses the x axis [9], [7].

There is only one equilibrium point, which is the double point on the boundary of $\mathcal{S}$ when $a = 1/16$. Also any trajectory asymptotic to the equilibrium point or a periodic solution which does not extend to infinity crosses the x axis after a finite time.

It has also been proven that no periodic solutions exist for $\underline{a}$ greater than 4 [6] because then $f < g$. A lot of specific results on periodic solutions have been obtained for the value $\gamma_1 = 0.97$ by Störmer [24], [26, 1950] and on a certain number of families of periodic solutions, the principal one [24], [13], [14], the oval family [3], [4], and the horseshoe family [9] and the family on the equator [5]. It will be seen now how the method of Part II leads very naturally to those results.

17. Surface of Section. The concept of surface of section, being especially useful to find periodic solutions, we will not consider the case $\gamma_1 \leqq 0$ where no such solutions exist; when $\gamma_1 > 0$ we have to distinguish three cases, we will treat first the most straightforward one.

i) $\gamma_1 > 1$ or $a < 0.0625$. It was proven by Graef that for these values of the parameter every trajectory crosses the line $\lambda = 0$ or is asymptotic to this line [12, p. 30, Th. VII and convention p. 28 Th. V]. Furthermore if a trajectory is asymptotic to $\lambda = 0$, the time between successive crossings is finite [7, Section 7], hence the segment PQ defined in the preceding section is a line of section (see also [7, Section 8]) for every trajectory of the region $\mathcal{S}_1$ of $\mathcal{S}$ not extending to $+\infty$. To $\mathcal{S}_1$ corresponds a connected bounded domain $\mathcal{T}$ inside the strip $g \leq x \leq g_1$ with boundary

$$\dot{x}^2 = 2U(x, 0, a) \quad . \tag{35}$$

Unfortunately the Theorem 11 can not be used, because $\mathcal{S}_1$ is not bounded and because the point at infinity is an essential singularity of U. We have to restrict $\mathcal{S}_1$, let us therefore use the coordinates of Lemaître-Bossy [16], obtained as follows: we draw from the point (x, λ) a

perpendicular to the thalweg (34), the distance to the thalweg is u, the latitude of the point on the perpendicular and the thalweg is y; the Jacobian of the transformation is positive and when $\lambda = 0$ we have $y = 0$, $x = u$ and $\dot{x} = \dot{u}$. Let us follow the trajectories in the space $u, \dot{u}, y$; the trajectories which are extremum ($\dot{y} = 0$) for a given value of $y = y_0$ are represented in the plane $y = y_0$ by points on a closed curve c_0 approximated by [16]:

$$\dot{u}^2 + (\cos y \cos v)^{-2} u^2 = a \cos^4 y$$

with $\tan v = 2 \tan y$. A point in the interior of c_0 corresponds to a trajectory having an extremum in y_0 greater than y_0 or passing through the origin. These trajectories will be represented on another plane $y = y_1 < y_0$ by curve c_1' interior to c_1 because every trajectory has only one extremum in y between each crossing of $y = 0$ [7]. For instance when $y = 0$, there exists a set of closed curves c_i' each one corresponds to trajectories having their extremum at $y = y_i > 0$ and c_i' is interior to c_j' if $y_i > y_j$. Let us now consider

$$\mathcal{S}_1' = \mathcal{S}_1 \cap (|y| \leqq y_i) \quad .$$

$\mathcal{S}_1'$ is a closed bounded domain in which U is analytic, if we exclude from $\mathcal{T}$ the domain inside c_i' and call this $\mathcal{T}_1'$, and define $\mathcal{T}_0' = R\mathcal{T}_1'$, to every point $p_0 \in \mathcal{T}_0'$ corresponds a unique solution of the differential equations (30) to (32) and this solution crosses again $\lambda = 0$ after a finite time at a corresponding point $P_1 \in \mathcal{T}_1'$ hence by Theorem 10 the correspondance is topological. Moreover, the part of the invariant curves $\mathcal{M}_1$ and $\mathcal{M}_2$ contained in $\mathcal{T}_1'$ is continuous. If it is true, as conjectured by Störmer [22, p. 145], that there is only one trajectory through the origin(1) and if for this trajectory the return is by convention the same path backwards, this will mean that the curves c_i' converge to the point 0 corresponding to the trajectory through the origin, when y_i tends to $\frac{\pi}{2}$. The transformation T should then be topological in $\mathcal{T}'$ and the $\mathcal{M}_n$ continuous. We will suppose this to be true in the following; if the conjecture is disproved, the results here stated will need an interpretation.

As no trajectory through the origin crosses the first time $\lambda = 0$ orthogonally, $\mathcal{M}_2$ does not pass through 0 and is analytic, $\mathcal{M}_1$ is made up of two branches corresponding to the two branches of the boundary of $\mathcal{S}_1$ which converge towards 0.

ii) $\gamma_1 = 1$ or $a = 0.0625$. This case is similar to the preceding one except for the fact that the boundary of $\mathcal{S}$ and $\mathcal{T}$ has a

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(1) In the following trajectory through the origin is by convention a trajectory through the origin in the r, λ coordinates, i.e., a trajectory passing through the essential singularity at infinity in the negative direction of the x axis.

double point, this does not alter any conclusion of i.

iii) $0 < \gamma_1 < 1$ or $a > 0.0625$. Here the situation is not as nice as in the preceding cases; in this case a trajectory starting from $\lambda = 0$ either crosses again $\lambda = 0$ or is unbounded, hence there does not exist a nice closed section of the $x, \dot{x}$ for which the transformation T transforms this section into itself. But the ideas of the second part of this paper may still be used because every periodic trajectory is bounded by the line $x = f$. Let us bound the domain inside of (35) by the line $x = f$ and call this $\mathcal{T}$; if $A \in \mathcal{T}$, TA is not necessarily in $\mathcal{T}$ or may even not exist if the trajectory goes to infinity before crossing $\lambda = 0$, but if A is periodic all the iterates of A are in $\mathcal{T}$; the invariant lines $\mathcal{M}_n$ are not entirely in $\mathcal{T}$ but their intersections $\mathcal{M}_{n,p}$ are necessarily in $\mathcal{T}$, hence to find the periodic solutions it is not necessary to know which part of $\mathcal{T}$ is transformed into itself but only to find the part of the iterates of $\mathcal{M}_0$ and $\mathcal{M}_1$ which is inside $\mathcal{T}$. This can be done in succession and we know that the iterate of a part of $\mathcal{M}_p$ not in $\mathcal{T}$ will not be in $\mathcal{T}$.

We will from here on summarize the content of the Report bearing the same title as this paper.

18. Data on the Computations. A sufficient number of points on the curves $\mathcal{M}_1$ and $\mathcal{M}_2$ have been determined for a sufficient number of values of the parameter a or γ_1. The choice has been made with great care and was dependent on preceding computation and on extensive computations done with a desk computer. The computations were done on a high-speed computer (the SEAC of the National Bureau of Standards). In the Report we have indicated clearly how the initial conditions were determined, which method of integration was used and which method of checking was involved. A total of 1288 trajectories were integrated for some 21 values of the parameter. The trajectories near $y = 0$ have also been integrated with another method to give the boundary points of $\mathcal{M}_1$ and $\mathcal{M}_2$. It may be interesting to note that the time for the integration of one step was 0.35 sec. for the high-speed computer versus 25 min. for the desk computer. For a precision of five to six decimal places, from 40 to 70 steps were necessary, in the mean.

19. Presentation of the Results. The results have been presented in the form of tables which are available on microfilm and in the form of diagrams, one for each value of the parameter γ_1; for $\gamma_1 = 0.97$, a detailed study has been made which leads to two diagrams, one of which is enclosed (Figure 1). Some data about Figure 1 will now be given.

The subspace $\mathcal{T}$ is bounded by the curve KJP and its symmetric.

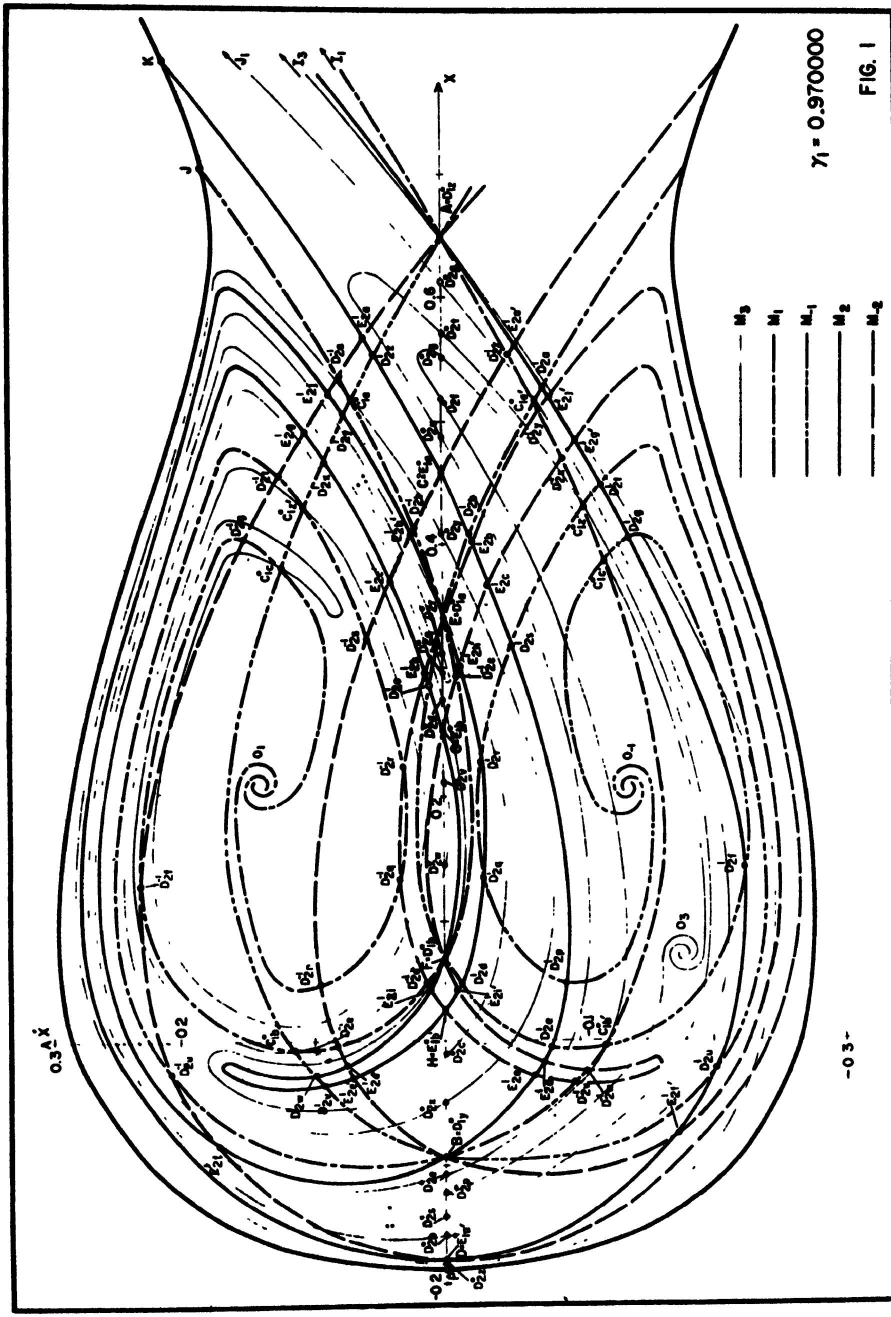

FIG. 1

The curves $\mathcal{M}_1$ and $\mathcal{M}_2$ have been determined by computation; each point corresponds to a trajectory with special initial conditions (Section 9). $\mathcal{M}_{-1}$ and $\mathcal{M}_{-2}$ have been deduced by symmetry. The two branches of $\mathcal{M}_1$ spiral towards a point O_1, which corresponds to a trajectory σ tending to the singularity $x = -\infty$, $\lambda = \frac{\pi}{2}$. The curves $\mathcal{M}_1$ have been deformed in the neighborhood of O_1 to simplify the drawing (of $\mathcal{M}_3$). The intersections of $\mathcal{M}_{-2}$, $\mathcal{M}_{-1}$, $\mathcal{M}_1$ and $\mathcal{M}_2$ give periodic points labeled according to their symmetry (Section 4); the letter subscript distinguishes different periodic solutions having the same symmetry. A to H correspond to periodic solutions which are particularly important.

The curve $\mathcal{M}_3$ has been obtained using only topological properties and its known intersections with the curves $\mathcal{M}_1$ and $\mathcal{M}_2$ deduced from Table I. Its drawing is inexact, the curve may be more complicated but not less complicated than what appears in Figure 1. Its intersections with the other curves and its symmetric (not drawn) give new symmetric periodic solutions. $\mathcal{M}_4$ has been constructed on another diagram. $\mathcal{M}_3$ has two branches which spiral towards a point O_3 which correspond to the second intersection of the trajectory σ with $\lambda = 0$.

20. Interpretation of the Results. From the diagrams it is possible to obtain results on the appearance of and disappearance of the symmetric periodic solutions. In particular, the orbits of type δ_1 and ϵ_1 have been studied (points A to H ...) and the results on their existence and stability, which had been obtained by special methods [3], [4], [14], [17] are deduced easily. New results include the study of the disappearance of orbits by passage to a limit position on $\lambda = 0$ related to [5], the proof of existence of many new periodic solutions which could hardly be obtained by the classical methods.

Analytic computations of the curves $\mathcal{M}_1$ and $\mathcal{M}_2$ which correspond to orbits close to the singular trajectory σ have been made using results of Lemaitre and Bossy [16] and the analytic computations of the same curves when γ_1 is large have also been done.

In one word the above method permits the unification of all the known results on Störmer's problem; at the same time it provides new results and new insight on the difficulties which one should expect for similar problems.

21. Remarks on the Classification. Before I conclude, I should like to make some remarks on the classification here given as applied to conservative problems. The reader will have no difficulty extending these remarks to any system of differential equations for which this classification can be used.

When a continuous variation of the parameter "a" was investigated, we saw that an orbit kept its classification except when the orbit disappeared; hence for a continuous variation of "a", the number of double points of the orbit on $y = 0$ remains the same. This is a special case of a more general theorem of Birkhoff [1, II, p. 60].

From this theorem we see that we can refine the classification by considering the double points of the orbit outside $y = 0$. It would be of interest to consider in more detail this subclassification which has, no doubt, a certain importance.

22. Conclusion. In the Introduction we have indicated the aims of this paper. We should like as conclusion, to bring out the relation of these investigations with connected research.

First of all, nothing has been gained here on problems such as transitivity or stability of systems of differential equations, but results are obtained which are much more detailed than those given by the general fixed point theorems, because not only is the existence of more than one or two periodic solutions proven, but the relations of the different periodic solutions with each other is obtained, i.e., the structure of the periodic solution is established.

There is a possibility that these investigations could lead to the proof, under certain hypotheses, of the density of the periodic solutions; in that case, every solution can be approximated during a finite time by a periodic solution.

The method here used has no essential limitations and enables dealing with problems with large non linearities as opposed to the perturbation methods which deal with small non linearities, but which generally fails otherwise (see, for instance, the application of perturbations method to Störmer's problem by G. Lemaitre [15]).

It is proper to mention that some methods (Van der Pol method and others) may succeed in special cases even with large non linearities.

Finally, we should like to mention that the results here obtained are not only of interest in themselves but because of their application to the region problems.

In these problems one considers two domains R_1, R_2 in the phase space and asks, for instance, the proportion of trajectories with initial conditions in R_1 that enter R_2. Solution to these problems depends on the knowledge of unstable periodic trajectories and trajectories asymptotic to these. Examples of this application are given for the cosmic rays' problem by Lemaitre, Vallarta, Bouckaert, Albagli Hunter, Yong-Li, and DeVogelaere (see references of [27]) and for a problem of chemical physics

(on transitions rates) by Boudart and DeVogelaere [10].

23. Acknowledgements. We dedicate this work to Odon Godart, Director of the Meteorological Service, Melsbroeck, Belgium. Conversations with him several years ago and notes on some of the many unpublished results he obtained just before the war served as inspiration for this research.

The preparatory desk computations were done by Marie-Paule Morisset under the sponsorship of Laval University, Quebec, Canada and of the National Research Council of Canada.

We also extend our thanks to John Todd of the National Bureau of Standards who made it possible for us to use the SEAC and to Allen Goldstein and especially Ida Rhodes who helped us in the coding and the running of the problem.

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V. ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF CANONICAL SYSTEMS NEAR A CLOSED, UNSTABLE ORBIT[1]

D. L. Slotnick*

§1. INTRODUCTION

We investigate in this paper the behavior of the solutions of a certain class of Hamiltonian systems with one degree of freedom, i.e., systems of the form

$$\dot{x} = H_y, \quad \dot{y} = - H_x$$

where the Hamiltonian $H(x, y, t)$ is a real analytic function of x and y, periodic in t, and vanishing at the origin $x = y = 0$. In the neighborhood of a closed orbit, autonomous Hamiltonian systems with two degrees of freedom can be reduced to this form, where the periodic orbit now corresponds to the zero solution. The historical interest in these systems rests largely on this ground.

We will deal with a class of these equations which have the feature of being unstable despite the stability of the linearized systems. For systems of this class we will establish the existence of an interesting family of unstable solutions and examine the analytic behavior of the surface on which these solutions escape from the origin. This behavior is marked by a rather unexpected phenomenon; namely while it would appear that this family should be analytic in some neighborhood of the origin it turns out to have an essential singularity at the origin. This has the immediate consequence of making power series methods inapplicable for obtaining these solutions.

In order to discuss the results in greater detail we give the

[1] This paper was submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy at New York University.

* This work was supported by the Office of Naval Research under contract No. N onr-285(06).

explicit form of the Hamiltonians under consideration:

$$H(x, y, t) = \frac{-\omega}{2}(x^2 + y^2) + H_3(x, y, t) + H_4(x, y, t) + \dots$$

is a convergent power series in some neighborhood of the origin. The terms $H_\sigma(x, y, t)$, $\sigma = 3, 4, 5, \dots,$ are homogeneous polynomials of degree σ in x and y whose coefficients are continuous functions of t with period 2π. ω is a positive constant.

The question of stability for these systems is still unsolved. Certainly the linear terms are stable but the answer for the full system is locked in the higher order terms. We denote by s the index for which $H_s(x, y, t) \not\equiv 0$ but $H_\sigma(x, y, t) \equiv 0$ for $\sigma < s$. Then we can write

$$H(x, y, t) = -\frac{\omega}{2}(x^2 + y^2) + H_s(x, y, t) + H_{s+1}(x, y, t) + \dots \quad .$$

An example of the dependency of the stability question on the higher order terms is furnished in [3]* where it is proved that under certain conditions on ω and on H_s -- related to its definiteness -- a strong growth condition holds for the solutions near the origin. Solutions satisfying this growth condition are called, by the author, <u>almost-stable</u>. Since we want to deal with unstable systems we have made a corresponding assumption of indefiniteness from which instability easily follows. This assumption is formulated in Section 3 after the Hamiltonian is brought into a simple normal form. In Sections 2 and 3 this normal form is obtained by exploiting the elegant algebraic properties of Hamiltonian systems. In particular the invariance of their form under canonical transformations. This allows one to concentrate completely on simplifying the Hamiltonian which, of course, then characterizes the corresponding system of differential equations.

In Section 4 we introduce, following Poincaré, the analytic mapping corresponding to the differential equations. This mapping assigns to a point (x_0, y_0) the new point $x_1 = x(2\pi)$, $y_1 = y(2\pi)$ where $x(t)$, $y(t)$ is the solution of the system with initial values x_0, y_0. Under the indefiniteness assumptions mentioned above we prove that this mapping has the form

$$\begin{aligned} x_1 &= s(x_0)x_0 + X(x_0, y_0) \\ y_1 &= t(x_0)y_0 + Y(x_0, y_0) \end{aligned} \tag{1.1}$$

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* Numbers in the square brackets [ ] refer to the bibliography at the end.

where $s(x_0) > 1 > t(x_0)$ for x_0 positive and $s(0) = t(0) = 1$. X and Y denote terms which in a suitable region are small compared with the leading terms.

This mapping is analogous in appearance with

$$(1.2) \qquad \begin{aligned} u_1 &= su + \dots \\ v_1 &= tv + \dots \quad , \end{aligned}$$

where s and t are constants which obey $s > 1 > t$ and the dots denote convergent power series in u, v which start with quadratic terms.

For the mapping (1.2) Poincare proved, by power series methods, the existence of an analytic curve which is invariant under the mapping [4]. Such a curve corresponds to a one-parameter family of solutions of the differential equations which move asymptotically away from the origin. Hadamard, assuming just the estimate $O(u^2 + v^2)$ and continuity for the higher order terms in (1.2) proved in [2], with a geometric argument, the existence of a continuous invariant curve. Recently, Sternberg [6] proved that the invariant curve for (1.2) is just as regular as the higher terms. Birkhoff [1] treated the mapping (1.1) for the case $s = 3$ and for this case, which is included in our result, proved the same result obtained here, by methods which are, however, completely different from those we employ and which are perhaps more complicated.

We prove in Section 5 that the mapping (1.1) possesses an invariant curve which is analytic for $x_0 > 0$ and is infinitely differentiable at $x_0 = 0$. We conclude by giving, in Section 6, an example adapted from [1] of such a cruve which has an essential singularity at $x_0 = 0$.

The author would like to thank his thesis adviser, Professor Jürgen Moser, for proposing the problem and for his generous help and encouragement throughout the time this paper was in preparation. Our thanks are due also to Professor K. O. Friedrichs for his kind cooperation.

§2. NORMALIZATION OF A HAMILTONIAN WITH A DEFINITE QUADRATIC TERM

This section and the next are devoted to showing that the periodic, analytic Hamiltonians we consider can be canonically transformed to an advantageous normal form. In this section we deal with those simplifications of the Hamiltonian that are a consequence of the assumed form of the quadratic term. In the next section we restrict our consideration to those Hamiltonians which are the subject of this paper by imposing conditions on

the next highest term and performing the further simplifications made possible by these additional assumptions. The reduction made in this section is specialized from one employed in [3].

We assume the Hamiltonian to be a real power-series, convergent in some neighborhood of $x = y = 0$, viz.:

$$(2.1) \quad H(x, y, t) = \frac{-\omega}{2}(x^2 + y^2) + H_s(x, y, t) + H_{s+1}(x, y, t) + \dots$$

where the subscript σ in $H_\sigma(x, y, t)$, here and throughout, denotes the degree of a homogeneous polynomial in x and y whose coefficients are continuous functions of t for all t and have period 2π. ω is a positive constant.

By introducing appropriate new variables ξ, η, we will show that as many terms of $H(x, y, t)$ as we desire, say $H_s, H_{s+1}, \dots, H_{s+n}$, can be transformed into a simple normal form. We want the system that results from this transformation to be again canonical and periodic in t, moreover since the quadratic term of $H(x, y, t)$ is already in a convenient form we will want the transformation to leave it unaltered. These ends can be accomplished by defining the transformation with a generating function of the form

$$(2.2) \quad u(x, \eta, t) = x\eta + u_s(x, \eta, t) + u_{s+1}(x, \eta, t) + \dots + u_{s+n}(x, \eta, t)$$

whereby the equations of the transformation are

$$\xi = u_\eta(x, \eta, t) = x + u_{s\eta} + u_{s+1\,\eta} + \dots + u_{s+n\;\eta}$$

$$y = u_x(x, \eta, t) = \eta + u_{sx} + u_{s+1\,x} + \dots + u_{s+n\;x}$$

and the Hamiltonian of the new system is readily seen to have the form

$$(2.3) \quad \Omega(\xi, \eta, t) = \frac{-\omega}{2}(\xi^2 + \eta^2) + \Omega_s(\xi, \eta, t) + \Omega_{s+1}(\xi, \eta, t) + \dots \;.$$

The coefficients of each $u_\sigma(x, \eta, t)$, $\sigma = s, s+1, \dots, s+n$, will be determined so as to have period 2π, thus insuring that $\Omega(\xi, \eta, t)$ also has this property, and so as to make $\Omega_s, \Omega_{s+1}, \dots, \Omega_{s+n}$ as simple as possible.

We carry this out first for $n = 0$, i.e., for the term $H_s(x, y, t)$. The Hamiltonian $\Omega(\xi, \eta, t)$ is, as is well known, related to $H(x, y, t)$ by

$$\Omega(u_\eta, \eta, t) = u_t + H(x, u_x, t)$$

which gives

$$\frac{-\omega}{2}(u_\eta^2 + \eta^2) + \Omega_s(u_\eta, \eta, t) + \ldots$$

$$= u_t - \frac{\omega}{2}(x^2 + u_x^2) + H_s(x, u_x, t) + \ldots \quad .$$

Comparing the terms of degree s in this last equation gives

$$(2.4) \qquad u_{st} + \omega(xu_{s\eta} - \eta u_{sx}) - \Omega_s(x, \eta, t) = - H_s(x, \eta, t) \quad ;$$

a partial differential equation for $u_s(x, \eta, t)$. It is convenient in discussing this equation to introduce the complex conjugate variables

$$\zeta = \xi + i\eta, \quad \bar{\zeta} = \xi - i\eta \quad .$$

The system in the variables $\zeta, \bar{\zeta}$ is canonical and its Hamiltonian

$$\Phi(\zeta, \bar{\zeta}, t) = - 2i\Omega\left(\frac{\zeta + \bar{\zeta}}{2}, \frac{\zeta - \bar{\zeta}}{2i}, t\right)$$

$$= i\omega\zeta\bar{\zeta} + \Phi_s(\zeta, \bar{\zeta}, t) + \Phi_{s+1}(\zeta, \bar{\zeta}, t) + \ldots \quad .$$

If we denote

$$u_s\left(\frac{\zeta + \bar{\zeta}}{2}, \frac{\zeta - \bar{\zeta}}{2i}, t\right)$$

by $v_s(\zeta, \bar{\zeta}, t)$ and replace x by ξ in (2.4) then (2.4) becomes

$$v_{st} + i\omega(\zeta v_{s\zeta} - \bar{\zeta} v_{s\bar{\zeta}}) - \frac{1}{2}\Phi_s(\zeta, \bar{\zeta}, t) = J_s(\zeta, \bar{\zeta}, t)$$

where $J_s(\zeta, \bar{\zeta}, t)$ is the known function

$$- H_s\left(\frac{\zeta + \bar{\zeta}}{2}, \frac{\zeta - \bar{\zeta}}{2i}, t\right) \quad .$$

Further, denoting the term in $\zeta^k\bar{\zeta}^\ell$, $k + \ell = s$, in v_s, Φ_s, J_s by $v_{k\ell}(t)\zeta^k\bar{\zeta}^\ell$, $\Phi_{k\ell}(t)\zeta^k\bar{\zeta}^\ell$, $J_{k\ell}\zeta^k\bar{\zeta}^\ell$ respectively then we see upon comparing the coefficients of $\zeta^k\bar{\zeta}^\ell$ in the last equation that

$$\dot{v}_{k\ell} + i\omega(k - \ell)v_{k\ell} - \frac{1}{2}\Phi_{k\ell} = J_{k\ell} \qquad \text{or}$$

(2.5)

$$\frac{d}{dt}\left(e^{i\omega(k-\ell)t}v_{k\ell}\right) - \frac{1}{2}e^{i\omega(k-\ell)t}\Phi_{k\ell} = e^{i\omega(k-\ell)t}J_{k\ell} \quad .$$

We would like to determine the $v_{k\ell}(t)$ so as to have period 2π and so as to make as many of the $\Phi_{k\ell}$ as possible zero. We claim this can be done for each k, ℓ for which $(k - \ell)\omega$ is not an integer. Thus if ω were irrational, for example, this could be done for all indices $k \neq \ell$. This can be proved as follows: set $\Phi_{k\ell} = 0$, then (2.5) becomes

$$\frac{d}{dt}\left(e^{i\omega(k-\ell)t}v_{k\ell}\right) = e^{i\omega(k-\ell)t}J_{k\ell}$$

and integrating from 0 to 2π gives

$$e^{2\pi i\omega(k-\ell)}v_{k\ell}(2\pi) - v_{k\ell}(0) = \int_0^{2\pi} e^{i\omega(k-\ell)t}J_{k\ell}(t)dt \quad .$$

Thus since $v_{k\ell}(t)$ has period 2π if and only if $v_{k\ell}(0) = v_{k\ell}(2\pi)$ we must choose

$$v_{k\ell}(0) = \left(e^{2\pi i\omega(k-\ell)} - 1\right)^{-1} \int_0^{2\pi} e^{i\omega(k-\ell)t}J_{k\ell}(t)dt$$

which can be done since the quantity in parentheses is not zero if $\omega(k - \ell)$ is not an integer. This proves that we can take $\Phi_{k\ell} = 0$ if $\omega(k - \ell)$ is not an integer. Observe that $\Phi_{k\ell}(t)$ has period 2π.

We now treat the remaining case: $\omega(k - \ell)$ an integer, which can occur only when $k = \ell$ or when ω is rational. We assert that in this case we can take

$$\Phi_{k\ell}(t) = \varphi_{k\ell}e^{-i\omega(k-\ell)t} , \tag{2.6}$$

where $\varphi_{k\ell}$ is a constant.

Substituting (2.6) into (2.5) and integrating from 0 to 2π gives

$$v_{k\ell}(2\pi) - v_{k\ell}(0) - i\pi\varphi_{k\ell} = \int_0^{2\pi} e^{i\omega(k-\ell)t}J_{k\ell}(t)dt \quad .$$

Thus $v_{k\ell}(0)$ will equal $v_{k\ell}(2\pi)$ independently of the value of $v_{k\ell}(0)$ if we choose

$$\varphi_{k\ell} = \frac{1}{\pi} \int_0^{2\pi} e^{i\omega(k-\ell)t}J_{k\ell}(t)dt \quad .$$

The function $v_s(\zeta, \bar{\zeta}, t)$ has thus been determined to have period 2π in t and so that the terms $\phi_{k\ell}(t)\zeta^k\bar{\zeta}^\ell$ of ϕ_s are different from zero only for those k, ℓ for which $(k - \ell)\omega$ is an integer and in this case we have $\phi_{k\ell}(t)$ given by (2.6). Now, if we choose $v_{\ell k}(0) = \bar{v}_{k\ell}(0)$ in the case $(k - \ell)\omega$ an integer it easily follows that $\bar{v}_{k\ell}(t) = v_{k\ell}(t)$ for all t and for all k, ℓ, so that $v_s(\zeta, \bar{\zeta}, t) = u_s(\xi, \eta, t)$ is real and since $\varphi_{\ell k} = -\bar{\varphi}_{k\ell}$, $\Phi(\zeta, \bar{\zeta}, t) = -2i\Omega$ is purely imaginary.

In the same way as u_s was chosen so that H_s was brought into the normal form given above we can now determine a function $u_{s+1}(x, \eta, t)$ such that the generating function

$$u(x, \eta, t) = x\eta + u_s + u_{s+1}$$

serves to carry H_s and H_{s+1} simultaneously into the normal form. This is so since the equation for u_{s+1} will just be (2.4) with s replaced by $s + 1$ on the left side and with the right side now depending on the known quantities H_s, H_{s+1}, Ω_s, and u_s. We can continue this process inductively arriving at a generating function

$$u = x\eta + u_s + u_{s+1} + \cdots + u_{s+n}$$

which simultaneously transforms H_s, H_{s+1}, ..., H_{s+n} into the normal form.

To summarize so far, the Hamiltonian (2.1) can without loss of generality be taken in the form

$$\Phi(\zeta, \bar{\zeta}, t) = i\omega\zeta\bar{\zeta} + \sum_{\sigma=s}^{s+n} \Phi_\sigma(\zeta, \bar{\zeta}, t) + \sum_{\tau=s+n+1}^{\infty} \Phi_\tau(\zeta, \bar{\zeta}, t)$$

with

$$\Phi_\sigma(\zeta, \bar{\zeta}, t) = \sum_{I_\sigma} \left(\varphi_{k\ell}^\sigma e^{i\omega(k-\ell)t}\zeta^k\bar{\zeta}^\ell - \bar{\varphi}_{k\ell}^\sigma e^{-i\omega(k-\ell)t}\zeta^\ell\bar{\zeta}^k\right) ,$$

$$\text{for } \sigma = s, s+1, \ldots, s+n,$$

where the index set I_σ is composed of those k, ℓ for which $k + \ell = \sigma$ and $(k - \ell)\omega$ is an integer.

We can now proceed to the final purpose of this section, namely to show that with a further canonical transformation, which is analytic and origin-preserving but not of period 2π, $\Phi(\zeta, \bar{\zeta}, t)$ can be transformed into a Hamiltonian which is time-independent through order $s + n$. We

define new variables z, $\bar{z}$ by

$$z = \zeta e^{-i\omega t}, \quad \bar{z} = \bar{\zeta} e^{i\omega t} \quad .$$

This canonical transformation is given by the generating function $\bar{\zeta} z e^{-i\omega t}$. The Hamiltonian of the transformed system is

$$\begin{aligned} \Lambda(z, \bar{z}, t) &= \Phi(z e^{i\omega t}, \bar{z} e^{-i\omega t}, t) - i\omega\zeta\bar{\zeta} \\ &= \sum_{\sigma=s}^{s+n} \Lambda_\sigma(z, \bar{z}) + \sum_{\tau=s+n+1}^{\infty} \Lambda_\tau(z, \bar{z}, t) \quad , \end{aligned} \tag{2.7}$$

where the terms

$$\Lambda_\sigma(z, \bar{z}) = \sum_{I_\sigma} \left(\varphi^\sigma_{k\ell} z^k \bar{z}^\ell - \bar{\varphi}^\sigma_{k\ell} z^\ell \bar{z}^k\right), \qquad \sigma = s, s+1, \ldots, s+n \quad ,$$

do not depend on t explicitly. This is the desired normal form.

We would of course like to go to the limit $n \longrightarrow \infty$ and thus obtain a time-independent Hamiltonian, which would be an integral of the system of equations, but the power series $x\eta + u_s + u_{s+1} + \cdots$ thus obtained is in general divergent as will be seen. It will suffice however for our purposes to know that this normalization can be carried out to any finite order.

In concluding this section we remark that this normalization shows that for the new Hamiltonian Λ the lowest order term is not $i\omega\zeta\bar{\zeta}$ any more but Λ_s and the nature of the motion therefore will be determined by Λ_s: $i\omega\zeta\bar{\zeta}$ being eliminated.

§3. FURTHER NORMALIZATION FOR THE CASE H_s INDEFINITE

We take the Hamiltonian in the form (2.7) and examine the special case $\Lambda = \Lambda_s(z, \bar{z})$. In this case $\Lambda_s(z, \bar{z})$ being a time-independent Hamiltonian is an integral of the system. If furthermore Λ_s is a definite form then by a well known theorem of Dirichlet the stability of the zero solution follows. Now since we will concern ourselves in this paper with a class of unstable systems we may assume that $\Lambda_s(z, \bar{z})$ is indefinite.

In order to formulate this more precisely we introduce polar coordinates ρ, ϑ into Λ_s by $z = \rho e^{i\vartheta}$. Since we will, for the moment, work only with the term Λ_s of Λ we omit the superscript s on the coefficients $\varphi^s_{k\ell}$ of Λ_s. Now, letting

$$\varphi_{k\ell} = |\varphi_{k\ell}| e^{i\tau_{k\ell}} \quad ,$$

an easy calculation shows that

$$\Lambda_s = 2i\rho^s \sum_{I_s} |\varphi_{k\ell}| \sin\left[(k - \ell)\vartheta + \tau_{k\ell}\right] = 2i\rho^s \sigma_s(\vartheta) \quad ,$$

where we have denoted the sum multiplying $2i\rho^s$ by $\sigma_s(\vartheta)$. Now if $\sigma_s(\vartheta)$ has a real zero it will ensure the indefiniteness of $\Lambda_s(z, \bar{z})$.

The requirement that Λ_s be indefinite is easily seen to be fulfilled by the following conditions on s and ω: ω equals a rational number $\frac{p}{q}$, in lowest terms, and both s and q are odd with $s \geq q$. For in this case $(k - \ell)\omega$ can be an integer if and only if $k - \ell$ is an odd multiple of q. Thus $\sigma_s(\vartheta)$ contains no constant term and hence has average value zero on 0 to 2π, which implies the existence of a real zero. Notice that for s even there is a term

$$\varphi_{\frac{s}{2}\frac{s}{2}}(z\bar{z})^{\frac{s}{2}}$$

in Λ_s giving rise to a constant term $\varphi_{\frac{s}{2}\frac{s}{2}}$ in $\sigma_s(\vartheta)$ which thus need not have a real zero.

> We henceforth assume that the Hamiltonian $H(x, y, t)$ has the property that $\sigma_s(\vartheta)$ has a real zero ϑ_0 and moreover that ϑ_0 is a simple zero.

In this section we will show what further simplifications of Λ can be made on the basis of this last assumption. Note that we can assume without loss of generality that the coordinates have been rotated so that $\vartheta_0 = 0$.

It is convenient to reintroduce real variables to (2.7) by

$$z = u + iv, \quad \bar{z} = u - iv \quad .$$

The Hamiltonian of the transformed system

$$Q(u, v, t) = \sum_{\sigma=s}^{s+n} Q_\sigma(u, v) + \sum_{\tau=s+n+1}^{\infty} Q_\tau(u, v, t)$$

with

$$Q_\sigma(u, v) = a_\sigma u^\sigma + b_\sigma u^{\sigma-1}v + \dots + e_\sigma v^\sigma; \quad \sigma = s, s+1, \dots, s+n \quad ,$$

where, in particular,

$$a_\sigma = - \sum_{I_\sigma} \operatorname{Im} \varphi^\sigma_{k\ell} \quad \text{and} \quad b_\sigma = - \sum_{I_\sigma} (k - \ell) \operatorname{Re} \varphi^\sigma_{k\ell} .$$

Now since we have assumed that 0 is a simple zero of $\sigma_s(\vartheta)$ it is clear that $Q_s(u, v)$ must contain v as a simple factor, i.e., that $a_s = 0$ while $b_s \neq 0$. The role of b_s is particularly important and for simplicity we will henceforth denote it by b.

We have thus shown that as a consequence of our last assumption $Q_s(u, v)$ must have v as a factor, i.e., that $a_s = 0$. In fact, we now prove that in appropriate new variables U, V, we can make V a factor of as many terms of the Hamiltonian as is desired. To this end we will determine a polynomial

$$P_{n+1}(u) = p_2 u^2 + p_3 u^3 + \dots + p_{n+1} u^{n+1}$$

with constant coefficients such that the canonical transformation

$$U = u, \qquad V = v + P_{n+1}(u)$$

of $Q(u, v\ t)$ yields a new Hamiltonian whose terms of degree $s + k + 1$ $(k = 0, 1, \dots, n-1)$ contain V as a factor, i.e., the value corresponding to a_{s+k+1} $(k = 0, 1, \dots, n-1)$ in the new Hamiltonian is zero.

PROOF. The transformed Hamiltonian is

$$Q\left[U, V - P_{n+1}(U)\right] = Q_s\left[U, V - P_{n+1}(U)\right] + \dots$$
$$+ Q_{s+k+1}\left[U, V - P_{n+1}(U)\right] + \dots .$$

One easily sees that the coefficient of U^{s+k+1} is

$$- bp_{k+2} + a_{s+k+1} + g_{s+k}$$

where g_{s+k} depends only on $p_2, p_3, \dots, p_{k+1}$ and the coefficients of $Q_s, Q_{s+1}, \dots, Q_{s+k}$. Thus if we consider $p_2, p_3, \dots, p_{k+1}$ as known by induction, we can choose

$$p_{k+2} = \frac{a_{s+k+1} + g_{s+k}}{b}$$

since $b \neq 0$, which concludes the proof.

To summarize, we have shown that in appropriate coordinates a Hamiltonian whose quadratic term is definite and whose next highest term is indefinite in the fashion described above can be written in the form

$$(3.2) \qquad Q(u, v, t) = \sum_{\sigma=s}^{s+n} Q_\sigma(u, v) + \sum_{\tau=s+n+1}^{\infty} Q_\tau(u, v, t) \quad ,$$

where $Q_\sigma(u, v) = b_\sigma u^{\sigma-1}v + \dots + e_\sigma v^\sigma$; $\sigma = s, s+1, \dots, s+n,$ and in particular, $b_s = b \neq 0$.

Here again as in the previous section we would like to go to the limit $n \longrightarrow \infty$ as then we would have v as a factor of Q. This would mean that if $u(t)$ were a solution of $\dot{u} = Q_v(u, 0, t)$ then $u = u(t)$, $v = 0$ would be a family of unstable solutions of the system derived from Q. This family would be of the type whose existence we seek to establish, with the curve $v = 0$ being the curve in question. However, we cannot let $n \longrightarrow \infty$ as the series $p_2u^2 + p_3u^3 + \dots$ thus obtained will in general diverge as will be seen from the counter-example of Section 6.

§4. EXISTENCE OF A CURVE INVARIANT UNDER THE ASSOCIATED MAPPING

In this section and the next we will utilize the normal form (3.2) to establish the behavior of the solutions of the differential equations arising from (3.2) in the vicinity of the solution $u = v = 0$. In this section we will actually need only the fact that Q_s and Q_{s+1} can be normalized as in (3.2). For the next section however we will need the full normalization to arbitrary order n.

We define the <u>weight</u> of the term u^kv^ℓ as $k + 2\ell$. The result of the last section then yields that

$$(4.1) \quad Q(u, v, t) = bu^{s-1}v + \text{terms of weight exceeding } s + 1, \; b \neq 0 \quad .$$

This is so because $a_s = a_{s+1} = 0$. It is precisely this consequence of the form (3.2) of Q that is important for this section in that if we confine u and v to the region R given by

$$(4.2) \qquad 0 \leq u < \rho, \quad |v| \leq u^2$$

where $Q(u, v, t)$ converges for $|u|, |v| < \rho$, then the fact that $bu^{s-1}v$ is the term of lowest weight in Q actually makes it the principal term of Q in R. This in turn will yield a desirable form for the principal terms of the mapping we will associate with the system. We now proceed to formulate

this explicitly.

The system arising from the Hamiltonian (4.1) has the form

$$\dot{u} = Q_v = bu^{s-1} + \text{terms of weight exceeding } s - 1$$

$$\dot{v} = - Q_u = - b(s - 1)u^{s-2}v + \text{terms of weight exceeding } s \quad .$$

If we next denote the initial values of a solution $u(t)$, $v(t)$ of the above system by u_0, v_0, it is a well known theorem, due to Cauchy, that the analyticity of the system implies that for some sufficiently small $\rho_1 > 0$, u and v are analytic functions of u_0, v_0 for $|u_0|$, $|v_0| < \rho_1$ and $0 \leq t \leq 2\pi$. Thus if we represent u and v as power series in u_0, v_0 with time-dependent coefficients, then substitution into the equation and comparison of coefficients, denoting $u(u_0, v_0, 2\pi)$, $v(u_0, v_0, 2\pi)$ by $u_1 = f(u_0, v_0)$, $v_1 = g(u_0, v_0)$, and setting $t = 2\pi$ gives

$$\begin{aligned} u_1 &= f(u, v) = u(1 + bu^{s-1}) + F(u, v) \\ v_1 &= g(u, v) = v(1 - b(s - 1)u^{s-2}) + G(u, v) \end{aligned} \tag{4.3}$$

where we have for simplicity denoted u_0, v_0, $2\pi b$ by u, v, b and where the power series $F(u, v)$, $G(u, v)$ begin with terms of weight exceeding $s - 1$: s respectively.

If we denote $1 + bu^{s-1}$, $1 - b(s - 1)u^{s-2}$ by $s(u)$, $t(u)$ respectively and make the assumption that $b > 0$, which we will later remove, then (4.3) has a form analogous to the mapping (studied by Poincaré):

$$u_1 = su + \ldots, \quad v_1 = tv + \ldots \tag{4.4}$$

where s and t are constants obeying $s > 1 > t$ and the dots denote quadratic and higher terms in u and v. The essential difference between (4.3) and (4.4) is that in (4.3) $s(u)$ and $t(u)$ while obeying $s(u) \geq 1 \geq t(u)$ for $u \geq 0$ however become equal to one for $u = 0$. This produces a striking difference in the properties of the two mappings namely that while it is proved in [4] that if (4.4) is analytic then it possesses an invariant curve passing through $u = v = 0$ which is itself analytic for $u \geq 0$, the invariant curve we shall prove exists for the analytic mapping given by (4.3) will in general have a singularity at the origin.

The analogy between (4.3) and (4.4) does hold insofar as in the region R the terms $s(u)u$, $t(u)v$ are the dominating terms in u_1, v_1 despite the fact that F and G will in general contain terms of the same degree as $s(u)u$ and $t(u)v$ respectively. This is precisely expressed in the following estimates: obtained in the usual way.

If (u, v) is in R with

$$u < \frac{\rho_1}{2} < \frac{1}{2}$$

then for a suitable constant M

$$|F(u, v)| \leq Mu^s, \quad |G(u, v)| \leq Mu^{s+1} \quad . \tag{4.5}$$

In addition, we list here for later use the following estimates:

$$|F_u| \leq Mu^{s-1}, \quad |F_v| \leq Mu^{s-2}, \quad |G_u| \leq Mu^s, \quad |G_v| \leq Mu^{s-1} \quad , \tag{4.6}$$

where M is taken large enough.

We now assert that

THEOREM 1. There exists a unique function $v = \varphi(u)$ lying in R with $\varphi(0) = 0$, invariant under the mapping (4.3).

Note that the image $\varphi_1(u)$ of a function $\varphi_0(u)$ under the mapping (4.3) is determined by the functional equation

$$g(u, \varphi_0(u)) = \varphi_1 \left[f(u, \varphi_0(u))\right] \quad . \tag{4.7}$$

LEMMA 1. If $|\varphi_0(u)| \leq u^2$, and $\varphi_1(u)$ is the image of $\varphi_0(u)$ under (4.3) then for

$$0 \leq u < \mathrm{Min}\left[\frac{b}{M}, \frac{1}{b(s-1)}\right], \qquad |\varphi_1(u)| \leq u^2 \quad ,$$

i.e., the image of a function in R is also in R.

PROOF. From (4.7),

$$\begin{aligned} |\varphi_1(u_1)| &= |\left(1 - b(s-1)u^{s-2}\right)\varphi_0(u) + G(u, \varphi_0)| \\ &\leq \left(1 - b(s-1)u^{s-2}\right)u^2 + Mu^{s-1} \end{aligned}$$

thus since

$$0 \leq u < \frac{1}{b(s-1)} \quad \text{and} \quad u < \frac{b}{M} < \frac{b(s-1)}{M}$$

we have

$$|\varphi_1(u_1)| \leq u^2 \quad .$$

Now, $u_1 = u + bu^{s-1} + F(u, \varphi_0) \geq u + bu^{s-1} - Mu^s$, hence we have $u_1 \geq u$, thus since $u < \rho_1 < 1$, we obtain

$$|\varphi_1(u_1)| \leq u^2 \leq u_1^2 \quad .$$

LEMMA 2. Let $\varphi_0(u)$ be as in Lemma 1. In addition assume $\varphi_0(u)$ is differentiable with $|\varphi_0'(u)| < \alpha$, where α is a positive constant with $\alpha < b(s - 1)/2M$. Then $\varphi_1(u)$, which is also differentiable, also obeys $|\varphi_1'(u)| < \alpha$ for $u < \mathrm{Min}[\alpha/s-1, 1/b(s - 1)]$, where $\frac{\alpha}{s-1} < \alpha$.

PROOF. Differentiating (4.7) gives

$$g_u + \varphi_0'(u)g_v = \varphi_1'(u_1)\left[f_u + \varphi_0'(u)f_v\right]$$

hence

$$|\varphi_1'(u_1)| = \left|\frac{g_u + \varphi_0'(u)g_v}{f_u + \varphi_0'(u)f_v}\right| \quad .$$

We estimate the denominator of this expression:

$$\begin{aligned}
f_u + \varphi_0'(u)f_v &= 1 + b(s - 1)u^{s-2} + F_u + \varphi_0'(u)F_v \\
&\geq 1 + b(s - 1)u^{s-2} - Mu^{s-1} - \alpha Mu^{s-2} \\
&= 1 + u^{s-2}[(b(s - 1) - \alpha M) - Mu] \\
&\geq 1 + u^{s-2}\left(\frac{b(s-1)}{2} - Mu\right) \geq 1 + u^{s-2}\left(\frac{b(s-1)}{2} - M\alpha\right) \geq 1 \quad .
\end{aligned}$$

The numerator in the expression for $|\varphi_1'(u_1)|$,

$$\begin{aligned}
&|g_u + \varphi_0'(u)g_v| \\
&= |- b(s - 1)(s - 2)u^{s-3}\varphi_0(u) + G_u + \varphi_0'(u)[1 - b(s - 1)u^{s-2} + G_v]| \\
&\leq b(s - 1)(s - 2)u^{s-1} + Mu^s + \alpha[1 - b(s - 1)u^{s-2} + Mu^{s-1}] \\
&= \alpha - u^{s-2}\{\alpha b(s - 1) - [b(s - 1)(s - 2) + M(\alpha + u)]u\} \\
&\leq \alpha - u^{s-2}\{\alpha b(s - 1) - [b(s - 1)(s - 2) + 2\alpha M]u\} \\
&\leq \alpha - u^{s-2}b(s - 1)\{\alpha - [(s - 2) + 1]u\} \\
&= \alpha - u^{s-2}b(s - 1)\{\alpha - (s - 1)u\} < \alpha \quad .
\end{aligned}$$

Combining the estimates for numerator and denominator we have $|\varphi_1'(u_1)| < \alpha$.

LEMMA 3. Let (u_1, v_1) be a point in R whose pre-image under (4.3) is (u, v). Here, (u, v) is to be thought of as a point on a curve in R and (u_1, v_1) as a point on the image curve. Then for sufficiently small positive u

$$\frac{1}{u^{s-2}} \leq \frac{1}{u_1^{s-2}} + 4b(s - 2) \quad .$$

PROOF.

$$u_1 \leq u(1 + bu^{s-2}) + Mu^{s-2}$$

$$= u[1 + u^{s-2}(b + Mu)], \quad \text{which for } u < \frac{b}{M} \text{ is}$$

$$\leq u(1 + 2bu^{s-2}) \quad .$$

Thus $u_1^{s-2} \leq u^{s-2}(1 + 2b(s - 2)u^{s-2} + \dots)$ which for

$$0 < u^{s-2} < \frac{1}{4b(s-2)}$$

is

$$\leq u^{s-2} \cdot \frac{1}{1 - 4b(s-2)u^{s-2}} \quad .$$

Whence

$$\frac{1}{u_1^{s-2}} \geq \frac{1}{u^{s-2}}\left(1 - 4b(s - 2)u^{s-2}\right) = \frac{1}{u^{s-2}} - 4b(s - 2) \quad .$$

We can now proceed to the proof of Theorem 1. The idea of this proof is adapted from [2] but is essentially modified.

PROOF OF THEOREM 1. Let $\varphi_0(u)$, $\tilde{\varphi}_0(u)$ be curves which have the properties being in R and obeying $|\varphi_0'|$, $|\tilde{\varphi}_0'| < \alpha$, then their iterates under (4.3) $\varphi_1, \tilde{\varphi}_1; \varphi_2, \tilde{\varphi}_2; \dots; \varphi_n, \tilde{\varphi}_n$ will, by Lemmas 1 and 2 also have these properties. We will prove that for sufficiently small u

$$\lim_{n \to \infty} |\varphi_n(u) - \tilde{\varphi}_n(u)| = 0$$

from which Theorem 1 will easily follow.

We begin by proving that

$$(4.8)\qquad |\varphi_{n+1}(u_1) - \tilde{\varphi}_{n+1}(u_1)| \leq \sigma(u)\ |\varphi_n(u) - \tilde{\varphi}_n(u)|$$

where

$$\sigma(u) = 1 - \frac{b(s-1)}{4}\, u^{s-2} \quad \text{and} \quad \sigma(u) > 0 \quad \text{for} \quad u < {}^1/b(s-1) \quad .$$

PROOF OF 4.8.

$$|\varphi_{n+1}(u_1) - \tilde{\varphi}_{n+1}(u_1)| \leq |\varphi_{n+1}(u_1) - \tilde{\varphi}_{n+1}(\tilde{u}_1)| + |\tilde{\varphi}_{n+1}(\tilde{u}_1) - \tilde{\varphi}_{n+1}(u_1)|$$

where $u_1 = f(u, \varphi_n(u))$ and $\tilde{u}_1 = f(u, \tilde{\varphi}_n(u))$; this last quantity is

$$\leq |g(u, \varphi_n(u)) - g(u, \tilde{\varphi}_n(u))| + |\tilde{\varphi}'_{n+1}|\ |f(u, \tilde{\varphi}_n(u)) - f(u, \varphi_n(u))|$$

$$\leq \Big\{|g_v| + \alpha|f_v|\Big\}\ |\varphi_n(u) - \tilde{\varphi}_n(u)|^{*}$$

$$= \Big\{1 - b(s-1)u^{s-2} + |G_v| + \alpha|F_v|\Big\}\ |\varphi_n(u) - \tilde{\varphi}_n(u)|$$

$$\leq \Big[1 - b(s-1)u^{s-2} + Mu^{s-1} + \alpha Mu^{s-2}\Big]\ |\varphi_n(u) - \tilde{\varphi}_n(u)|$$

$$= \Big[1 - u^{s-2}(b(s-1) - \alpha M) + Mu^{s-1}\Big]\ |\varphi_n(u) - \tilde{\varphi}_n(u)|$$

$$\leq \Big[1 - u^{s-2}\Big(\frac{b(s-1)}{2} - Mu\Big)\Big]\ |\varphi_n(u) - \tilde{\varphi}_n(u)|$$

which for

$$u < \frac{b(s-1)}{4M}$$

is

$$\leq \Big(1 - \frac{b(s-1)}{4}\, u^{s-2}\Big)\ |\varphi_n(u) - \tilde{\varphi}_n(u)| = \sigma(u)\ |\varphi_n(u) - \tilde{\varphi}_n(u)| \quad ,$$

where

$$\sigma(u) = 1 - \frac{b(s-1)}{4}\, u^{s-2} \quad .$$

Now, if we denote by $(u_{-1}, \varphi_{n-1}(u_{-1}))$ the pre-image of the point $(u, \varphi_n(u))$ and in general denote the pre-image of $(u_{-k}, \varphi_{n-k}(u_{-k}))$ by $(u_{-k-1}, \varphi_{n-k-1}(u_{-k-1}))$ we can continue (4.8) as follows:

---

* In the brackets { } v must be replaced by a mean value between $\varphi_n(u)$ and $\tilde{\varphi}_n(u)$ before applying (4.6).

$$|\varphi_{n+1}(u_1) - \tilde{\varphi}_{n+1}(u_1)| \leq \sigma(u)\sigma(u_{-1})\ |\varphi_{n-1}(u_{-1}) - \tilde{\varphi}_{n-1}(u_{-1})|$$

$$\leq \sigma(u)\sigma(u_{-1})\sigma(u_{-2})\ \cdots\ \sigma(u_{-n})\ |\varphi_o(u_{-n}) - \tilde{\varphi}_o(u_{-n})| \quad .$$

Now, since φ_n was obtained as the $\underline{nth}$ iterate of a curve $\varphi(u)$ in R we know that $(u_{-n}, \varphi_o(u_{-n}))$ lies in R; thus we know that

$$|\varphi_o(u_{-n}) - \tilde{\varphi}_o(u_{-n})| < 2\rho_1^2 = \mu \quad .$$

Substituting this into the last inequality above gives

$$|\varphi_{n+1}(u_1) - \tilde{\varphi}_{n+1}(u_1)| \leq \sigma(u)\sigma(u_{-1})\sigma(u_{-2})\ \cdots\ \sigma(u_{-n})\mu \quad . \tag{4.9}$$

To conclude (4.8) from (4.9) we prove that the product on the right in (4.9) approaches zero as n approaches infinity.

If the

$$u_{-k}^{s-2} < c, \quad \text{where} \quad c = \frac{1}{4b(s-2)}$$

we see upon repeated application of Lemma 3 that

$$\frac{1}{u_{-k}^{s-2}} \leq \frac{1}{u_{-k+1}^{s-2}} + \frac{1}{c} \leq \frac{1}{u^{s-2}} + \frac{k}{c}, \quad \text{for} \quad k = 1, 2, \ldots, n \quad .$$

Thus

$$u_{-k}^{s-2} \geq \frac{u^{s-2}}{1 + u^{s-2}kc^{-1}} = c\left[\frac{1}{u^{s-2}c^{-1}} + k\right] \geq \frac{c}{\kappa + k}$$

where κ is, for instance, the first positive integer greater than

$$\frac{c}{u^{s-2}} \quad .$$

Using this last obtained inequality gives

$$\sigma(u_{-k}) \leq \left(1 - \frac{s-1}{s-2}\,\frac{1}{16}\,\frac{1}{\kappa + k}\right) \leq \left(1 - \frac{1}{16}\,\frac{1}{\kappa + k}\right) \quad .$$

Thus from (4.9) we have

$$|\varphi_{n+1}(u_1) - \tilde{\varphi}_{n+1}(u_1)|$$

$$\leq \left(1 - \frac{1}{16\kappa}\right)\left(1 - \frac{1}{16(\kappa+1)}\right)\left(1 - \frac{1}{16(\kappa+2)}\right)\ \cdots\ \left(1 - \frac{1}{16(\kappa+n)}\right)\mu$$

Recalling the formula

$$\frac{1}{\Gamma(z)} = \lim_{m \longrightarrow \infty} m^{1-z} \prod_{n=1}^{m} \left(1 + \frac{z-1}{n}\right)$$

we see that the last quantity above approaches zero like

$$\frac{\text{constant}}{n^{1/16}}$$

which proves (4.8).

If now in the above we consider $\tilde{\varphi}_0$ to be the image of φ_0 and, generally, denote the image of φ_{n-1} by φ_n then (4.8) implies that for each sufficiently small $u > 0$, $\{\varphi_n(u)\}$ is a Cauchy sequence, say $\varphi_n(u) \longrightarrow \varphi(u)$. Then $\varphi(u)$ is unique. It is to show this uniqueness that we considered the iterates of two curves; to show simply existence, the iterates of a single curve were in fact used. $\varphi(u)$ is the single-valued invariant function we seek. That the values of $\varphi(u)$ form a curve will be proved in the next section where, of course, we shall prove more than this.

In concluding this section we remark that if in Lemma 3 we choose (u, v) to be on the invariant curve then so will (u_1, v_1) and the inequality demonstrates that the solutions of the differential equations which have initial values on the invariant curve are not semi-stable in the sense of [3] since the u_{-n} approach the origin as rapidly as

$$n^{\frac{-1}{s-2}} \quad .$$

§5. PROPERTIES OF THE INVARIANT CURVE

In this section we establish the following theorem concerning the invariant function $\varphi(u)$ whose existence and uniqueness were proved in the last section.

THEOREM 2. The function $\varphi(u)$ is

(a) analytic for sufficiently small $u > 0$, and

(b) infinitely differentiable at $u = 0$.

In Section 6 we will show by a counter-example, which is a modification of one used by Birkhoff in [1], that $\varphi(u)$ need not be analytic at $u = 0$ so that Theorem 2 is a best possible result.

PROOF OF (a). We assume $Q(u, v, t)$ to be in the form (4.1), i.e., normalized through order $s + 1$ and in (4.3) we consider the variables u, v, u_1, v_1 to be complex. We will show that we can determine a region in the u-plane and a sequence of analytic functions $\{\varphi_n(u)\}$ convergent in this region, and whose limit function is $\varphi(u)$ when u is real, which satisfy the hypotheses of Vitali's Theorem.

To this end let S be a circular sector in the u-plane, symmetric with respect to the positive real u-axis, whose angle ϑ_o and radius $\rho_1 < 1$ we will appropriately determine. Let $v = \varphi_o(u)$ be an analytic function which obeys $|\varphi_o(u)| \leq |u|^2$ for u in S and is real for real values of u. The image of S under φ_o will then lie within the circle $C\colon |v| \leq \rho_1$. Now, consider $f(u, \varphi_o(u))$, $g(u, \varphi_o(u))$. As u ranges over S these will sweep out regions in the u_1 and v_1 planes; denote these regions by S_1, C_1 respectively. We assert that by properly choosing ϑ_o and ρ_1 we can achieve

$$S_1 \supset S \quad \text{and} \quad C_1 \subset C \quad , \tag{5.1}$$

(a) will follow directly from (5.1) as we shall see.

We prove now the first part of (5.1). We assert firstly that for all ϑ_o such that

$$0 < \vartheta_o \leq \frac{\pi}{4(s-2)}$$

there exists a positive real number $K(\vartheta_o)$ such that $\arg u_1 \geq \vartheta_o$ for all u of argument equal to ϑ_o and modulus less than $K(\vartheta_o)$.

This last assertion can be proved as follows: We have from (4.3) that

$$u_1 = u(1 + bu^{s-2}) + F(u, \varphi_o(u)) = u\left\{1 + bu^{s-2} + \frac{F(u,\varphi_o)}{u}\right\} \quad .$$

Thus taking the $\arg u = \vartheta_o$ we obtain

$$\log \rho_1 + i\vartheta_1 = \log \rho + i\vartheta_o + \log \left\{1 + bu^{s-2} + \frac{F(u,\varphi_o)}{u}\right\} \quad .$$

Equating imaginary parts gives

$$\begin{aligned}
\vartheta_1 &= \vartheta_o + \operatorname{Im} \log \left\{1 + bu^{s-2} + \frac{F(u,\varphi_o)}{u}\right\} \\
&= \vartheta_o + \operatorname{Im} \left\{\left(bu^{s-2} + \frac{F}{u}\right) - \frac{1}{2}\left(bu^{s-2} + \frac{F}{u}\right)^2 \right. \\
&\qquad\qquad \left. + \frac{1}{3}\left(bu^{s-2} + \frac{F}{u}\right)^3 + \dots\right\} \\
&= \vartheta_o + b|u|^{s-2} \sin(s - 2)\vartheta_o + \text{terms containing } |u|^{s-1}
\end{aligned}$$

We can thus determine $K(\vartheta_0)$ such that if $|u| < K(\vartheta_0)$ this last quantity is

$$\geq \vartheta_0 + \frac{b}{2}\,|u|^{s-2}\sin(s-2)\vartheta_0 \geq \vartheta_0 + \frac{a}{\pi}\,|u|^{s-2} \geq \vartheta_0 \quad .$$

Note that $K(\vartheta_0)$ goes to zero with ϑ_0.

To conclude the proof of the first part of (5.1) we easily see that if $\vartheta < \vartheta_0$ and

$$|u| < \frac{b}{2M}$$

then $|u_1| \geq |u|$ since

$$\begin{aligned} |u_1| &\geq |u|\ |1 + bu^{s-2}| - M|u|^s \\ &\geq |u|\ \left(1 + \frac{|bu^{s-2}|}{2}\right) - M|u|^s \\ &= |u|\left[1 + |u^{s-2}|\left(\frac{b}{2} - M|u|\right)\right] \geq |u| \quad . \end{aligned}$$

Thus choosing

$$\vartheta_0 < \frac{\pi}{4(s-2)}$$

and

$$\rho_1 < \min\left[1, \frac{b}{2M}, K(\vartheta_0)\right]$$

insures that the first part of (5.1) holds. We claim that this also serves to insure the second part of (5.1) for

$$|u| < \frac{1}{b(s-1)} \quad .$$

PROOF. Denote by $\varphi_1(u)$ the image under (4.3) of $\varphi_0(u)$. Then by (4.7) we have

$$\varphi_1(u_1) = \varphi_0(u)(1 - b(s-1)u^{s-2}) + G(u, \varphi_0(u)) \quad .$$

Dividing by u^2 and taking moduli we get

$$\begin{aligned} \left|\frac{\varphi_1(u_1)}{u^2}\right| &\leq \left|\frac{\varphi_0(u)}{u^2}\right|\ |1 - b(s-1)u^{s-2}| + \left|\frac{G}{u^2}\right| \\ &\leq |1 - b(s-1)u^{s-2}| + M|u|^{s-1} \end{aligned}$$

$$\le 1 - \frac{b(s-1)}{2} |u|^{s-2} + M|u|^{s-1}$$

$$= 1 - |u|^{s-2} \left(\frac{b(s-1)}{2} - M|u|\right) \le 1$$

since

$$|u| < \frac{b}{2M} < \frac{b(s-1)}{2M} \quad .$$

Thus since $|u| \le |u_1| \le \rho_1$ we have $|\varphi_1(u_1)| \le \rho_1^2 < \rho_1$ since $\rho_1 < 1$. This concludes the proof of (5.1).

We now complete the proof of (a). From the analyticity of (4.3) we know that $\varphi_1(u)$ inherits the analyticity of $\varphi_0(u)$. $\varphi_1(u)$ is defined in the region S_1 which by (5.1) contains S. Thus $\varphi_1(u)$ is defined in S, and its values lie within $C_1 \subset C$. In addition $\varphi_1(u)$ will

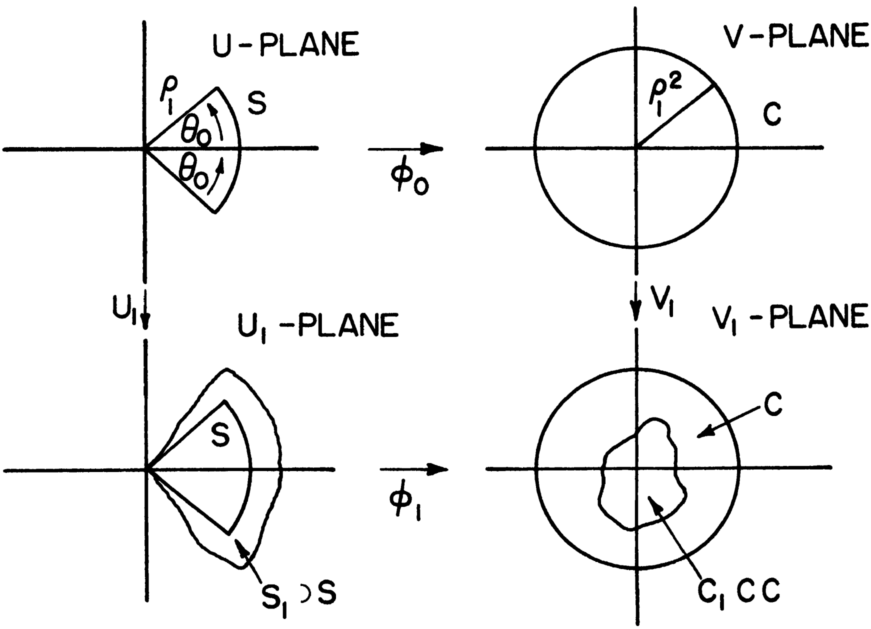

FIGURE 1

be real for real values of u. Thus $\varphi_1(u)$ will have all of the properties of $\varphi_0(u)$. See Figure 1.

Continuing in this manner yields a sequence $\varphi_0(u)$, $\varphi_1(u)$, $\varphi_2(u)$, ... which by (5.1) are bounded in S and by Theorem 1 converge to $\varphi(u)$ for real values of u. Vitali's Theorem then tells us that the sequence converges to a function analytic in the interior of S, thus no information is provided at $u = 0$ which is on the boundary of S. Specializing this result to real values of u gives precisely (a).

PROOF OF (b). The proof of (b) is carried out by showing that $\varphi(u)$ has a unique asymptotic expansion at $u = 0$.

Recall that the Hamiltonian can be taken in the normal form (3.2). This can be rephrased in terms of the mapping as follows: By a transformation

$$U = u, \qquad V = v + P_n(u)$$

$$U_1 = u_1, \qquad V_1 = v_1 + P_n(u_1)$$

(4.3) becomes a mapping with V_1 of the form

$$V_1 = V(1 - b(s - 1)U^{s-2}) + V\,W_n(U, V) + Z_N(U, V) \tag{5.2}$$

where W_N is a polynomial whose lowest and highest terms have degree $s - 1$ and $N + s - 2$ respectively and Z_N has degree exceeding $N + s - 2$.

We assert that the

$$P_N(u) = \sum_{k=2}^{N} p_k u^k$$

above satisfy

$$\lim_{u \to 0} \left| \frac{\varphi(u) + P_{N-2}(u)}{u^{N-1}} \right| = -\,p_{N-1} \quad ,$$

i.e., that

$$-\sum_{k=2}^{\infty} p_k u^k$$

is the asymptotic expansion of the invariant function $\varphi(u)$ at $u = 0$.

PROOF. Denote the invariant function written in the U, V coordinates by $\Psi(U)$; then clearly

$$\varphi(u) = - P_N(u) + \Psi(u) \quad ,$$

hence

$$\varphi(u) + P_{N-2}(u) = - p_{N-1}u^{N-1} - p_N u^N + \Psi(u) = R_{N-2}(u)$$

and our assertion reduces to proving that

$$\lim_{u \to 0} \left| \frac{R_{N-2}(u)}{u^{N-1}} \right| = - p_{N-1} \quad .$$

This we do as follows: Let $\Psi_0(U)$ be a function obeying $|\Psi_0(U)| \leq U^N$ and $\Psi_1(U)$ be its image under the mapping in the form (5.2). From the functional equation (4.7) and from (5.2) we have

$$\Psi_1(U_1) = \Psi_0(U)(1 - b(s - 1)U^{s-2}) + \Psi_0(U)\, W_N(U, \Psi_0(U)) + Z_N \quad .$$

Thus

$$\left| \frac{\Psi_1(U_1)}{U^N} \right| \leq \left| \frac{\Psi_0(U)}{U^N} \right| \left(1 - b(s - 1)U^{s-2} + |W_N|\right) + \frac{|Z_N|}{U^N}$$

which for suitable constants M_1 and M_2 is

$$\leq 1 - b(s - 1)U^{s-2} + M_1 U^{s-1} + M_2 U^{s-1}$$

which for

$$0 \leq U \leq \operatorname{Min} \left[\frac{1}{b(s-1)}, \frac{b(s-1)}{M_1 + M_2} \right] \quad \text{is} \quad \leq 1 \quad .$$

Thus since we can clearly choose U sufficiently small to insure that $U_1 \geq U$ we have

$$|\Psi_1(U_1)| \leq U_1^N \quad .$$

Continuing in this way we obtain that all the iterates $\Psi_0, \Psi_1, \Psi_2, \cdots$ have this property, hence $|\Psi(U)| \leq U^N$.

Now this last result together with the fact that

$$\frac{R_{N-2}(u)}{u^{N-1}} = - p_{N-1} - p_N u + \frac{\Psi(u)}{u^{N-1}}$$

clearly shows that

$$\lim_{u \to 0} \left| \frac{R_{N-2}(u)}{u^{N-1}} \right| = - p_{N-1} ,$$

which concludes the proof of (b).

We can now remark that the entire argument of Sections 4 and 5 can be carried out for any simple real zero ϑ_0 of $\sigma_s(\vartheta)$. Notice also that with ϑ_0, $\vartheta_0 + \pi$ is a root of $\sigma_s(\vartheta)$, thus the invariant curves come in pairs which are tangent to the same line at the origin but these pairs are not the analytic continuation of each other because of the singularity at the origin.

Also, notice that the transformation yielding (2.7) has the period $\frac{2\pi}{q}$ in t, where $\omega = \frac{p}{q}$. Thus in the original coordinates the invariant curve has the following significance. A solution $x(t)$, $y(t)$ with initial values on the curve returns to the curve after the time $2q\pi$ at a point closer to the origin. Thus after a time $t = 2q\pi$ the entire curve $\varphi(u)$ will return to its position at $t = 0$. For values of t between 0 and $2q\pi$ the curve will sweep out some surface whose plane sections perpendicular to the t-axis are indicated in Figure 2.

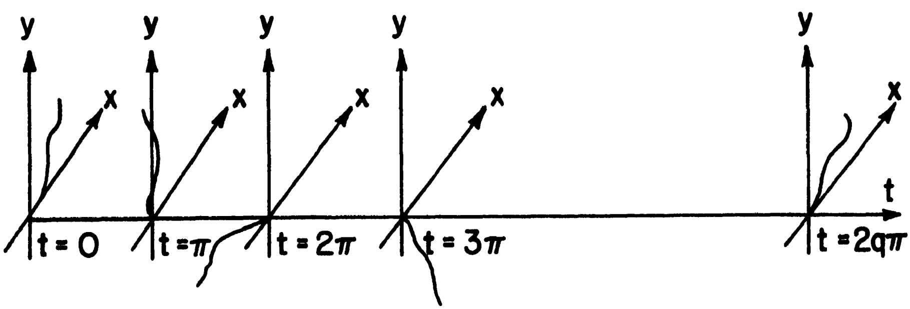

FIGURE 2

Finally we can remove the restriction that the constant $b > 0$ by remarking that if we consider the inverse mapping of (4.3) then the role of b is played by $-b$, so that we can reduce the case $b < 0$ to the one we have treated by working in this case with the inverse mapping.

§6. COUNTEREXAMPLE TO ANALYTICITY AT $u = 0$

We now give the counterexample to analyticity of $\varphi(u)$ at $u = 0$ mentioned at the beginning of Section 5. We will first have to recall some properties of the function

$$\Psi(z) = \frac{d}{dz} \log \Gamma(Z) \quad ,$$

where $\Gamma(z)$ is the gamma function.

$\Gamma(z)$ obeys $\Gamma(z + 1) = z\Gamma(z)$; taking logarithms and differentiating 4 times gives

$$\Psi'''(z + 1) = -\frac{6}{z^4} + \Psi'''(z) \quad . \tag{6.1}$$

It is this functional equation for $\Psi'''(z)$ that we will have need for.

We take as our mapping

$$\begin{aligned} u_1 &= \frac{u}{(1+u^{s-2})^{\frac{1}{s-2}}} = f(u) \\ v_1 &= (1 + u^{s-2})^{\frac{s-1}{s-2}} (v - 6u^{3s-7}) = g(u, v) \end{aligned} \tag{6.2}$$

which an easy calculation shows to be area preserving and of the form (4.3) when considered expanded into its power series. First, we make the useful observation that

$$\frac{1}{u_1^{s-2}} = \frac{1}{u^{s-2}} + 1 \quad . \tag{6.3}$$

Now we claim that the curve given by

$$\varphi(u) = \frac{1}{u^{s-1}} \Psi''' \left(\frac{1}{u^{s-2}}\right) \tag{6.4}$$

is left invariant under (6.2).

PROOF. We must show that

$$g(u, \varphi(u)) = \varphi(f(u)) \quad . \tag{6.5}$$

This will be done by reducing (6.5) to the functional equation (6.1). We write (6.5) in extenso:

$$(1 + u^{s-2})^{\frac{s-1}{s-2}} \left\{\frac{1}{u^{s-1}} \psi''' \left(\frac{1}{u^{s-2}}\right) - 6u^{3s-7}\right\}$$

$$= \frac{(1 + u^{s-2})^{\frac{s-1}{s-2}}}{u^{s-1}} \psi''' \left(\frac{1}{u_1^{s-2}}\right)$$

Using (6.3) in the right-hand side and performing some elementary manipulation reduces this to

$$\psi''' \left(\frac{1}{u^{s-2}} + 1\right) = \psi''' \left(\frac{1}{u^{s-2}}\right) - 6u^{4(s-2)} .$$

Finally, setting

$$\frac{1}{u^{s-2}} = z$$

gives exactly (6.1).

To conclude that this is the desired counterexample requires only to remark that $\varphi(u)$, as defined, is analytic within a circular sector about the positive real u-axis of angle

$$< \frac{2\pi}{s - 2} ,$$

is real for real u, and has an essential singularity at $u = 0$: it has there a divergent asymptotic representation, see, for example, [1] and [5].

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VI. SMALL PERIODIC PERTURBATIONS OF AN AUTONOMOUS SYSTEM OF VECTOR EQUATIONS*

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INTRODUCTION

In a study on non-linear analytic ordinary differential equations printed in 1934, N. M. Kryloff and N. N. Bogoliuboff [1] published the first results in perturbation theory where a set of solutions was the primary object of study. They established the existence of a two dimensional torus generated by solutions whose initial values formed a cross section. This bounded set of solutions is a natural generalization of the one dimensional torus, i.e., the periodic solution.

The cross section of initial values can be interpreted as a curve that is invariant under a transformation induced by the differential equation. The existence of a torus of solutions follows immediately from the existence of an invariant curve.

N. Levinson published a similar result in 1950 [2]. He removed the restriction of analyticity and required that the unperturbed equation have a strongly stable periodic solution. Using a geometric interpretation of Levinson's proof as a starting point, S. P. Diliberto [3], [7], proposed a perturbation theory where the objects studied are periodic k-surfaces. (A surface generated by solutions to a system of differential equations is called a periodic k-surface if it is the homeomorphic image of the product of k circles.) I made a study of a class of transformations that are used in this theory and obtained sufficient conditions for the existence of an invariant surface. My result was published in Volume Three of this series [4].

The purpose of this paper is to apply this theorem to a perturbation problem closely related to that of Kryloff and Bogoliuboff. Their

* This report represents results obtained at the Institute of Mathematical Sciences, New York University, under the auspices of the U. S. Army Office of Ordnance Research, Contract No. DA-30-069-ORD-1258.

** Now with Shell Development Company, Emeryville, California.

torus of solutions is a periodic 2-surface. In the problem studied here, a set of periodic $(k + 1)$-surfaces is shown to exist. The members of this set are generated by solutions starting on periodic k-surfaces.

SECTION 1

Using vector notation, a system of n differential equations can be written

$$\frac{dx}{dt} = X(x) \quad . \tag{1}$$

We assume that the vector valued function X has continuous third partial derivatives. We also assume that the system of differential equations has a periodic solution $x = u(t)$ of period ω.

In the (x, t) space, the range of this periodic solution is a closed curve in the $t = 0$ plane. Let us call it β_0. The solutions to (1) whose initial values lie on β_0 are periodic solutions, $x = u(t + t_0)$. These solutions generate a cylinder parallel to the t-axis. By the uniqueness theorem, a solution once touching this surface must remain on it for all t.

Now consider a perturbed system

$$\frac{dx}{dt} = X(x,\ t,\ \gamma) \quad . \tag{2}$$

We assume that the third partial derivatives of X with respect to x and t are continuous in x, t, and γ, and are periodic in t with period T. γ is a real parameter. At $\gamma = 0$, the system (2) becomes the autonomous system (1).

In the $t = 0$ plane, let β be a smooth simple closed curve <u>near</u> the curve defined by the periodic solution to (1). If γ is small, we can use the points of the curve β as sets of initial values for solutions to the perturbed system. A surface will be generated whose intersection with the plane $t = T$ will be a simple closed curve. In this way a transformation is defined on a class of curves. If it is possible to find a curve that is fixed under this transformation, then the solutions starting on its image in the $t = T$ plane will trace out the same curve in the $t = mT$ plane, for any integer m. This follows from the periodicity of the perturbed system. Therefore if, for some γ, an invariant curve exists, the solutions to (2) that start on it will generate a periodic surface in the (x, t) space. By identifying cross-sections at $t = 0$ and $t = T$, this surface becomes a two dimensional torus in n space. The generating curves are solutions to a differential equation on this torus.

It is easy to see that such invariant curves usually will not exist. It is therefore necessary to place further restrictions on the unperturbed equations (1). Levinson [2] assumed that the periodic solution $x = u(t)$ was strongly stable. This means that the linear equation of first variation

$$\frac{dv}{dt} = \sum_{j=1}^{n} \frac{\partial X(u(t))}{\partial x_j} v_j \tag{3}$$

has $n - 1$ characteristic exponents with negative real parts. He then proved that for each γ sufficiently small, there is a unique invariant curve β_γ, depending continuously on γ, and having the property that as γ tends to zero, β_γ approaches β_0, the curve determined by the periodic solution to (1). In the (x, t) space, as γ tends to zero, the periodic surfaces defined by these invariant curves collapse onto the cylinder defined by the periodic solution.

The hypothesis about the characteristic exponents is similar to the non-vanishing Jacobian hypothesis of the implicit function theorem. Diliberto conjectured that it would be sufficient if the $n - 1$ characteristic exponents have non-zero real parts. This insures the invertibility of an operator corresponding, in a sense, to the Jacobian. In 1953, G. Hufford [5] proved Diliberto's conjecture.

Let us consider the following generalization of this problem. Take a system of vector equations

$$\frac{dx_i}{dt} = X_i(x_i) \qquad i = 1, \ldots, k \quad , \tag{4}$$

where for each i, we have an n dimensional vector equation. Each vector equation is assumed to have a periodic solution with period ω_i (the ω_i may be unequal). In the kn dimensional space formed by taking Cartesian products, these solutions define a k dimensional torus. The solutions to (4) starting on this k-torus generate a cylinder parallel to the t-axis.

The perturbed system

$$\frac{dx_i}{dt} = X_i(x_1, x_2, \ldots, x_k, t, \gamma) \tag{5}$$

introduces coupling (and periodic time dependence, with period T) that disappears as γ tends to zero. We take as a set of initial points a k dimensional surface <u>near</u> the k-torus. For small γ, a transformation is then defined by following solutions until they intersect the $t = T$ plane.

The generalized problem is to find sufficient conditions so that for each γ sufficiently small, there is a unique invariant surface. This invariant surface will define a periodic surface in the (x, t) space that can be considered as a $(k + 1)$ dimensional torus in a $(kn + 1)$ dimensional space. As γ tends to zero, this must collapse onto the original surface.

The first results that apply to the generalized problem were obtained by M. Marcus [6] in 1953. His theorem, when specialized to $k = 1$, includes Levinson's but not Hufford's. By using a fixed point theorem that is stated in Section 3 (and proved in [4]), I have shown that an invariant k-torus exists, if, for each i, the unperturbed vector equation (5) satisfies Hufford's hypothesis.

SECTION 2

In order to prove the existence of an invariant surface, we make use of a coordinate transformation introduced by Levinson for the $k = 1$ problem. For each i, we can transform the corresponding n dimensional space to coordinates $(y_i^1, y_i^2, \ldots, y_i^{n-1}, \theta_i)$ by

$$x_i = u_i(\theta_i) + S_i(\theta_i)y_i \quad . \tag{6}$$

$S_i(\theta_i)$ is an $n \times (n-1)$ matrix whose entries are C^4 periodic functions of the single variable θ_i with period ω_i. u_i is the periodic solution of the unperturbed i-th vector equation. This transformation is valid near the curve defined by the range of u_i. It induces a transformation of the product space.

In this coordinate system, the periodic solution u_i is given by $y_i = 0$, $\theta_i = t$. The system of vector equations (4) can be written

$$\begin{aligned} \frac{dy_i}{dt} &= Y_i(y_i, \theta_i) \\ \frac{d\theta_i}{dt} &= \Theta_i(y_i, \theta_i) \end{aligned} \tag{7}$$

where Y_i and Θ_i have period ω_i in θ_i. $Y_i(0, \theta_i) \equiv 0$, $\Theta_i(0, \theta_i) \equiv 1$.

The perturbed system becomes

$$\begin{aligned} \frac{dy_i}{dt} &= Y_i(y_1, \ldots, y_k, \theta_1, \ldots, \theta_k, t, \gamma) \\ \frac{d\theta_i}{dt} &= \Theta_i(y_1, \ldots, y_k, \theta_1, \ldots, \theta_k, t, \gamma) \quad . \end{aligned} \tag{8}$$

Y_i and Θ_i have period ω_i in θ_i and period T in t. As γ becomes zero, (8) becomes (7).

Consider a solution to the perturbed system that starts from a point in the $t = 0$ plane with coordinates $(y, \theta) = (y_1, \ldots, y_k, \theta_1, \ldots, \theta_k)$. (Recall that (y_i, θ_i) is a point in the n dimensional space of x_i.) This solution can be written

$$\text{(9)} \qquad \begin{aligned} \bar{y} &= F(t, y, \theta, \gamma) \\ \bar{\theta} &= G(t, y, \theta, \gamma) \quad . \end{aligned}$$

Since $\bar{y} = 0$, $\bar{\theta} = t + \theta$ is the solution of $\gamma = 0$, $y = 0$, $\theta = \theta$, we can write (9) as

$$\text{(10)} \qquad \begin{aligned} \bar{y} &= N(\theta)y + R(y, \theta, \gamma) \\ \bar{\theta} &= \theta + T + H(y, \theta, \gamma) \quad , \end{aligned}$$

where the solution is evaluated at $t = t$. $N(\theta)$ is the $k(n-1) \times k(n-1)$ matrix formed by the partial derivatives of the vector function F with respect to y evaluated at $\gamma = 0$, $y = 0$, $\theta = \theta$. R is a $k(n-1)$ vector, H a k vector. Both are C^2 functions of y and θ, with period ω_i in θ_i.

At $y = 0$, $\gamma = 0$, we have

$$\text{(11)} \qquad \begin{aligned} R(0, \theta, 0) &\equiv 0 \quad , \\ \frac{\partial R(0,\theta,0)}{\partial y} &\equiv 0 \quad , \\ H(0, \theta, 0) &\equiv 0 \quad . \end{aligned}$$

Equation (10) defines a mapping of the point (y, θ) in the $t = 0$ plane onto the point $(\bar{y}, \bar{\theta})$ in the $t = T$ plane. This induces a map of surfaces onto surfaces by taking $y = \beta(\theta_1, \ldots, \theta_k)$, where β has period ω_i in θ_i and is of class C^2. The image of β is a surface π in the $t = T$ plane. π is given by

$$\text{(12)} \qquad \begin{aligned} \pi(\bar{\theta}) &= N(\theta)\beta(\theta) + R(\beta(\theta), \theta, \gamma) \\ \bar{\theta} &= \theta + T + H(\beta(\theta), \theta, \gamma) \quad . \end{aligned}$$

To compute the value of π at $\bar{\theta}$, it is necessary to solve the second equation of (12) for θ as a function of β, $\bar{\theta}$, and γ. Geometrically, we are finding the θ coordinate of the point $(\pi(\bar{\theta}), \bar{\theta})$ in the $t = T$ plane by going back along a solution curve to the point $(\beta(\theta), \theta)$ in the $t = 0$ plane. This can be done if the norms of β (see below) and of γ are small enough.

Let Q_γ be the transformation such that $\pi = Q_\gamma(\beta)$. This transformation of surfaces into surfaces can be considered as a non-linear transformation defined on a subset of a space of multiply periodic functions. Our problem is to show that this transformation has a fixed point.

SECTION 3

If we let $m = k(n - 1)$, our function space is the space of real multiply periodic continuously differentiable m-dimensional vector valued functions of k real variables. If $\beta = (\beta_1, \ldots, \beta_m)$ is such a function, then β must be periodic with period ω_i in θ_i.

We use

$$\|\beta\| = \max\left[\|\beta\|_0, \left\|\frac{\partial\beta}{\partial\theta_i}\right\|_0\right] ,$$

where

$$\|\beta\| = \max_j \max_\theta |\beta_j(\theta)| .$$

For fixed γ, a transformation $\pi = Q_\gamma(\beta)$ is given by equations (12). To insure that $\bar{\theta}$ can be solved for θ, we restrict $\|\beta\|$ and $|\gamma|$ so that the matrix

$$\left(\frac{\partial\bar{\theta}_j}{\partial\theta_t}\right)$$

is non-singular.

If $\|N\| < 1$, it is possible to work with Q_γ directly and pick a bounded equi-continuous set that is mapped into itself. This set can be selected so that Q_γ is continuous on it. By Schauder's theorem, there is a fixed point in the set. This method was used by Marcus.

However if we do not know that $\|N\| < 1$, (e.g., $k = 1$, Hufford's theorem) this method cannot be used. It is convenient to do the following: If β is a fixed point, i.e., $\beta = Q_\gamma(\beta)$, then the first equation of (12) can be written

$$\beta(\bar{\theta}) - N(\theta)\beta(\theta) = R(\beta(\theta), \theta, \gamma) \quad . \tag{13}$$

For fixed β and γ, a linear transformation $L_{\beta\gamma}$ is defined on the function space by

$$\left[L_{\beta\gamma}\rho\right](\theta) = \rho(\bar{\theta}) - N(\theta)\rho(\theta)$$

(14)

$$\bar{\theta} = \theta + T + H(\beta(\theta), \theta, \gamma) \quad .$$

If $L_{\beta\gamma}$ is invertible, then equation (13) becomes

$$\beta = L_{\beta\gamma}^{-1}R(\beta, \gamma) \quad , \tag{15}$$

where $[R(\beta, \gamma)](\theta) = R(\beta(\theta), \theta, \gamma)$. Because of the big O properties of R (see eq. (11)), it is possible to apply Schauder's theorem to the transformation $L^{-1}R$.

In [4], I studied transformations of the type Q_γ and showed that if $L_{\beta\gamma}^{-1}$ is uniformly bounded in norm, there exists $\gamma_0 > 0$, such that if $|\gamma| \leq \gamma_0$, Q_γ has a unique fixed point β_γ. β_γ is continuous in γ, and as $\gamma \longrightarrow 0$, $\beta_\gamma \longrightarrow 0$. Furthermore

$$\left\{ \frac{\partial \beta_\gamma}{\partial \theta_t} \right\}$$

satisfy a uniform Lipschitz condition. (Recall that the 0 of the function space corresponds to the k-torus defined by the periodic solutions to the unperturbed differential equations (4)).

SECTION 4

PROPOSITION. If for each i, $i = 1, 2, \ldots, k$, the vector equation (4) has a linear equation of first variation

$$\frac{dv}{dt} = \sum_{j=1}^{n} \frac{\partial X_i(u_i(t))}{\partial x_j} \tag{16}$$

with $n - 1$ characteristic exponents with non-zero real parts, then for suitably restricted β and γ, $L_{\beta\gamma}^{-1}$ exists and is uniformly bounded in the norm.

PROOF. The matrix $N(\theta)$ is found by taking the partial derivatives of $F(t, y, \theta, \gamma)$ with respect to y_i. Since the derivatives are evaluated at $\gamma = 0$, the effect of coupling disappears. Therefore

$$(17) \qquad N(\theta) = \begin{pmatrix} N_1(\theta_1) & & & 0 \\ & N_2(\theta_2) & & \\ & & \ddots & \\ 0 & & & N_k(\theta_k) \end{pmatrix} .$$

If $\bar{y}_i = F_i(t, y_i, \theta_i)$ is the solution to the unperturbed (and therefore uncoupled) i-th equation, we have

$$(18) \qquad N_i(\theta_i) = \frac{\partial F_i(T, 0, \theta_i)}{\partial y} .$$

Because of the special form of $N(\theta)$, it is sufficient to study the transformation defined on a subspace of $(n-1)$-vector valued functions given by

$$\left[L^i_{\beta\gamma}\pi\right](\theta_1, \theta_2, \theta_3, \ldots, \theta_k) = \pi(\bar{\theta}_1, \ldots, \bar{\theta}_k) - N_i(\theta_i)\pi(\theta_1, \ldots, \theta_k)$$

(19)

$$\bar{\theta} = G_{\beta\gamma}(\theta) = \theta + T + H(\beta(\theta), \theta, \gamma) .$$

π is an $(n-1)$-vector, β a $k(n-1)$-vector, and θ is a k-vector.

We will take $L_{\beta\gamma}$ as a transformation on the space of <u>complex</u> valued functions. If $L^{-1}_{\beta\gamma}$ exists, it induces a transformation on the space of real valued functions.

In an appendix, it is shown that

$$(20) \qquad N_i(\theta_i) = P(\theta_i + T)JP^{-1}(\theta_i) .$$

P is a non-singular periodic matrix, J is a constant matrix in canonical form, i.e.,

$$(21) \qquad J = \begin{pmatrix} J_1 & & 0 \\ & \ddots & \\ 0 & & J_t \end{pmatrix} \qquad J_v = \begin{pmatrix} \lambda_v & 0 & 0 & \ldots & 0 \\ 1 & \lambda_v & 0 & \ldots & 0 \\ 0 & 1 & \lambda_v & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & & \ldots & \lambda_v \end{pmatrix} , \; v = 1, \ldots, t.$$

Since the characteristic exponents of (16) are the logarithms of the characteristic roots of J, we have that $|\lambda_v| \neq 1$.

We now define linear transformations on a subspace of $(n-1)$-vectors by

$$
\begin{aligned}
[\bar{P}\pi](\theta) &= P(\theta)\pi(\theta) \\
[T^{-1}\pi](\theta) &= \pi(\theta - T) \\
[\bar{G}_{\beta\gamma}\pi](\theta) &= \pi(G_{\beta\gamma}(\theta)) \\
[\bar{J}\pi](\theta) &= J\pi(\theta) \quad .
\end{aligned} \tag{22}
$$

Then

$$
T^{-1}L^{1}_{\beta\gamma} = T^{-1}\bar{G}_{\beta\gamma} - \bar{P}\bar{J}T^{-1}\bar{P}^{-1} = \bar{P}(\bar{P}^{-1}T^{-1}\bar{G}_{\beta\gamma}\bar{P} - \bar{J}T^{-1})\bar{P}^{-1} \quad . \tag{23}
$$

Let

$$
K_{\beta\gamma} = \bar{P}^{-1}T^{-1}\bar{G}_{\beta\gamma}\bar{P} - \bar{J}T^{-1} \quad . \tag{24}
$$

$$
\left[K_{\beta\gamma}\pi\right](\theta) = P^{-1}(\theta)P\Big(G_{\beta\gamma}(\theta - T)\Big)\pi\Big(G_{\beta\gamma}(\theta - T)\Big) - J\pi(\theta - T) \tag{25}
$$

$$
G_{\beta\gamma}(\theta - T) = \theta + H(\beta(\theta - T), \theta - T, \gamma) \quad . \tag{26}
$$

Because of this, if

$$
P_{\beta\gamma}(\theta) = P\Big(G_{\beta\gamma}(\theta)\Big) - P(\theta) \quad , \tag{27}
$$

then (using the big O properties of H) $\|P_{\beta\gamma}\|$ can be made small by restricting $\|\beta\|$ and $|\gamma|$. We can write equation (25) as

$$
\left[K_{\beta\gamma}\pi\right](\theta) = \pi\Big(G_{\beta\gamma}(\theta - T)\Big) - J\pi(\theta - T) + P^{-1}(\theta)P_{\beta\gamma}(\theta)\pi\Big(G_{\beta\gamma}(\theta - T)\Big) \tag{28}
$$

The first two terms define a transformation $M_{\beta\gamma}$ that acts independently on the subspaces determined by the distinct characteristic values of J. The last term can be made small in norm.

Let

$$
\left[M^{v}_{\beta\gamma}\phi\right](\theta) = \phi\Big(G_{\beta\gamma}(\theta - T) - J_v\phi(\theta - T)\Big) \quad . \tag{29}
$$

ϕ is a complex r-dimensional vector valued function, where r is the

multiplicity of λ_v. If $M^v_{\beta\gamma}$ has a uniformly bounded inverse, then $M_{\beta\gamma}$ will. By using the standard series technique, this implies that $K_{\beta\gamma}$, and therefore $L^1_{\beta\gamma}$, and finally $L_{\beta\gamma}$ have uniformly bounded inverses.

For fixed β and γ, $\bar{\theta} = G_{\beta\gamma}(\theta)$ differs only slightly from the identity map. We can solve for θ to get $\theta = V_{\beta\gamma}(\bar{\theta})$. Define the linear transformation $\bar{V}_{\beta\gamma}$. Then

(30) $$M^v_{\beta\gamma} = T^{-1}\bar{G}_{\beta\gamma}\left(I - \bar{V}_{\beta\gamma}\bar{J}_v\right) .$$

(31) $$\left[I - \bar{V}_{\beta\gamma}\bar{J}_v\right]\phi(\theta) = \phi(\theta) - J_v\phi\left(V_{\beta\gamma}(\theta)\right) .$$

To show that this transformation has a uniformly bounded inverse, we make a very slight modification of the proof used by Hufford in Lemma 3.1 of his thesis.

If $|\lambda_v| < 1$, take δ so that $|\lambda_v| + \delta < 1$. Let $\bar{S}$ be defined by the matrix

(32) $$S = \begin{pmatrix} \delta & 0 & 0 & \cdots & 0 \\ 0 & \delta^2 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & \delta^r \end{pmatrix}$$

Then

(33) $$SJ_vS^{-1} = \begin{pmatrix} \lambda_v & 0 & 0 \cdots & 0 & 0 \\ \delta & \lambda_v & 0 \cdots & 0 & 0 \\ \cdot & \cdot & \cdot \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot \cdots & \cdot & \cdot \\ 0 & 0 & 0 \cdots & \delta & \lambda_v \end{pmatrix}.$$

This matrix defines a linear transformation $\bar{S}\,\bar{J}_v\bar{S}^{-1}$. $\|\bar{S}\,\bar{J}_v\bar{S}^{-1}\| = |\lambda_v| + \delta$.

(34) $$\left(I - \bar{V}_{\beta\gamma}\bar{J}_v\right) = \bar{S}^{-1}\left(I - \bar{V}_{\beta\gamma}\bar{S}\,\bar{J}_v\bar{S}^{-1}\,\bar{S}^{-1}\right)\bar{S} .$$

$\|\bar{V}_{\beta\gamma}\| \leq 1 + q$, where q can be made arbitrarily small (see appendix). Therefore $\|\bar{V}_{\beta\gamma}\bar{S}\,\bar{J}_v\bar{S}^{-1}\| \leq (1 + q)(|\lambda_v| + \delta) < 1$, if q is properly chosen.

$$(35)\qquad \left\|\left(I - \bar{V}_{\beta\gamma}\bar{S}\,\bar{J}_v\bar{S}^{-1}\right)^{-1}\right\| = \left\|\sum_{u=0}^{\infty}\left[\bar{V}_{\beta\gamma}\bar{S}\,\bar{J}_v\bar{S}^{-1}\right]^u\right\| \le (1/q) \quad .$$

q is independent of β and γ (if $\|\beta\| \le r_0$, $|\gamma| \le \gamma_0$), hence we have a uniform bound.

If $|\lambda_v| > 1$, then J_v^{-1} has the characteristic root $(1/\lambda_v) < 1$, and

$$(36)\qquad \left(I - \bar{V}_{\beta\gamma}\bar{J}_v\right)^{-1} = -\left(I - \bar{G}_{\beta\gamma}\bar{J}^{-1}\right)^{-1}\bar{G}_{\beta\gamma}\bar{J}^{-1} \quad .$$

$\|\bar{G}_{\beta\gamma}\| \le 1 + q'$, q' small. We again have a uniform bound.

Retracing our steps, we see that $L_{\beta\gamma}^{-1}$ as a transformation on a space of complex valued functions is uniformly bounded. The transformation induced by it on the space of real valued function has the same bound. This completes the proof of the proposition.

CONCLUDING REMARKS

My fixed point theorem was applied to a special situation where the existence of a periodic coordinate transformation and the existence of L^{-1} depended on their existence in the single variable case. If the unperturbed periodic surface of initial values is not the product of periodic solutions, it is not yet possible to show that the differential equations (5) can be transformed into the system (8) so that the original periodic surface is given by $y = 0$, $\theta = \theta$.

In the general k variable problem the concept of characteristic exponents is not applicable. A method of characterizing the unperturbed system of differential equations is needed so that one could easily tell if L^{-1} exists and is uniformly bounded. Diliberto has recently obtained such a result for the special case of $m = 1$ [page 5.3 of [7]].

APPENDIX

PROPOSITION. $N_1(\theta_1) = P(\theta_1 + T)J\,P^{-1}(\theta_1)$.

PROOF. Dropping subscripts, we have that $\bar{y} = F(t, y, \theta)$, $\bar{\theta} = G(t, y, \theta)$ is the solution of

$$(1)\qquad \begin{aligned} \frac{dy}{dt} &= Y(y, \theta) \\ \frac{d\theta}{dt} &= \Theta(y, \theta) \end{aligned}$$

that starts at (y, θ) in the $t = 0$ plane. Since $\bar{y} = 0$, $\bar{\theta} = T + \theta$, is a solution, $Y(0, \theta) \equiv 0$, $\Theta(0, \theta) \equiv 1$. The equation of first variation is

$$\frac{d\bar{w}}{dt} = \begin{pmatrix} \dfrac{\partial Y(0,t+\theta)}{\partial y} & 0 \\ \dfrac{\partial \Theta(0,t+\theta)}{\partial \theta} & 0 \end{pmatrix} \bar{w} \quad . \tag{2}$$

$\dfrac{\partial Y}{\partial y}$ is an $(n-1) \times (n-1)$ matrix.

$$\frac{d}{dt}\left[\frac{\partial F(t,0,\theta)}{\partial y}\right] = \frac{\partial Y(0,t+\theta)}{\partial y} \frac{\partial F(t,0,\theta)}{\partial y} \quad . \tag{3}$$

If $W(t)$ is a matrix of solutions to

$$\frac{dW}{dt} = \frac{\partial Y(0,t)}{\partial y} W \tag{4}$$

such that $W(0) = I$, then

$$\frac{\partial F(t-\theta,0,\theta)}{\partial y} = W(t)W^{-1}(\theta) \quad .$$

Set $t = \theta + T$, then

$$N(\theta) = \frac{\partial F(T,0,\theta)}{\partial y} = W(\theta + T)W^{-1}(\theta) \quad . \tag{5}$$

Equation (4) is a linear differential equation with periodic coefficients. Its characteristic exponents are the $(n - 1)$ characteristic exponents with non-zero real parts associated with the original variational equation.

Therefore there exists a periodic matrix $Q(t) = Q(t + \omega_1)$, $Q(0) = I$, and a constant matrix A such that

$$W(t) = Q(t) \exp[tA] \quad . \tag{6}$$

Let $\exp[tA] = RJR^{-1}$, $P(\theta) = Q(\theta)R$, then equation (5) becomes

$$N(\theta) = P(\theta + T)JP^{-1}(\theta) \quad . \tag{7}$$

This proves the proposition.

PROPOSITION. Given $q' > 0$, there exist r_0, γ_0 so that if $\|\beta\| \leq r_0$, and $|\gamma| \leq \gamma_0$, then $\|\bar{G}_{\beta\gamma}\| \leq 1 + q'$.

PROOF.

(8)
$$\left[\bar{G}_{\beta\gamma}\pi\right](\theta) = \pi\Big(G_{\beta\gamma}(\theta)\Big) \quad .$$

Since $\bar{\theta} = G_{\beta\gamma}(\theta) = \theta + T + H(\beta(\theta), \theta, \gamma)$ is one to one,

(9)
$$\left\|\bar{G}_{\beta\gamma}\pi\right\|_0 = \max_i \max_\theta \left|\pi_i\Big(G_{\beta\gamma}(\theta)\Big)\right| = \max_i \max_\theta \left|\pi_i(\theta)\right| = \left\|\pi\right\|_0 \quad .$$

(10)
$$\frac{\partial\pi(G_{\beta\gamma}(\theta))}{\partial\theta_t} = \sum_{i=1}^{k} \frac{\partial\pi(G_{\beta\gamma}(\theta))}{\partial\theta_i}\left[\delta_{ij} + \sum_{j=1}^{m} \frac{\partial H_i(\beta(\theta),\theta,\gamma)}{\partial y_j}\,\frac{\partial\beta_j}{\partial\theta_t} + \frac{\partial H_i(\beta(\theta),\theta,\gamma)}{\partial\theta_t}\right] .$$

$\frac{\partial H}{\partial y}$ is uniformly bounded, and $H(0, \theta, 0) \equiv 0$. Hence we can take $\|\beta\|$ and $|\gamma|$ small enough to insure that

(11)
$$\left|\frac{\partial\pi(G_{\beta\gamma}(\theta))}{\partial\theta_t}\right| \leq \left\|\left|\frac{\partial\pi}{\partial\theta_t}\right|\right\|_0 [1 + q'] \quad .$$

Therefore

(12)
$$\|\bar{G}_{\beta\gamma}\pi\| \leq \|\pi\|\ [1 + q'] \quad .$$

Having this, we can show that the inverse map, $\bar{V}_{\beta\gamma}$ satisfies

(13)
$$\|\bar{V}_{\beta\gamma}\pi\| \leq \|\pi\|\ [1 + q] \quad .$$

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VII. ROTATED VECTOR FIELDS AND AN EQUATION FOR RELAXATION OSCILLATIONS

George Seifert[1]

1. INTRODUCTION

We are concerned with systems of differential equations of the form:

$$(1.1) \qquad \dot{x} = P(x, y, \alpha), \quad \dot{y} = Q(x, y, \alpha) ;$$

here the dots stand for differentiation with respect to the independent variable t, and α is a real parameter. We shall refer to the curves in an (x, y) cartesian plane, the so-called phase plane, as the phase trajectories, or simply, the trajectories, of the system (1.1). The critical points of (1.1), the points of the phase plane for which P and Q are both zero, while corresponding to solutions of this system, will not be referred to as trajectories, are assumed independent of α, and constitute a set of points having no finite limit point in the phase plane. Ordinary points are points which are not critical.

At each ordinary point (x, y) a vector having components $(P(x, y, \alpha), Q(x, y, \alpha))$ is defined, and for fixed α the totality of all such vectors will be called the vector field of (1.1) corresponding to that value of α. If all the vectors of field experience a rotation in the same sense as α is varied monotonically in some interval, we say that (1.1) defines rotated vector fields for α on this interval. G. F. D. Duff [1] has studied a type of rotated vector fields largely for the purpose of determining the behavior of limit cycles, i.e., trajectories which consist of simple closed curves and correspond to periodic solutions of (1.1), in terms of variations in α.

It is the purpose of this paper to discuss a more general type of

[1] This research was supported in part by the United States Air Force through the Office of Scientific Research of the Air Research and Development Command.

rotated vector field, primarily for the purpose of determining the behavior of limit cycles in terms of variations in α. It is found that if two half-trajectories, corresponding to distinct values of α and originating at the same point, intersect at a point beyond the initial point, the region enclosed must contain a critical point. A non-intersection theorem for cycles corresponding to distinct values of α is obtained. A theorem concerning the monotonic and continuous variation of asymptotically stable (which we will henceforth refer to as simply stable) limit cycles follows. Instead of a theorem such as Duff obtains concerning the nature of the bounds of the annular regions covered by varying cycles, we prove a theorem of somewhat similar content, largely, however, for the purpose of proving an existence theorem for periodic solutions of a general equation for relaxation oscillations. In particular, the equation $\ddot{x} + f(x, \dot{x})\dot{x} + g(x) = 0$ is considered, and a set of conditions on f and g are given under which a periodic solution exists. The special case for which

$$f(x, y) = \frac{x^4}{|y| + 1} - 1, \quad g(x) = x/2,$$

to which the existence criterion given by Levinson and Smith [2] cannot be applied, satisfies these conditions. In fact, it seems that the requirement given in [2] that there exist an $x_0 > 0$ such that for $|x| > x_0$ and all y, $f(x, y) > 0$ can be relaxed to the extent that for each fixed y we need $f(x, y) > 0$ for x sufficiently large, provided the dependence of f on x is large enough in comparison with that of g.

2. GENERAL RESULTS

We assume that P and Q of (1.1) are continuous in (x, y, α), satisfy Lipschitz condition with respect to x and y in any bounded domain of the (x, y) plane, and that for a domain $\mathcal{D}$ of this plane and a real interval $\mathcal{J}$, the following conditions are satisfied:

(A) the partial derivatives P_α and Q_α exist at each ordinary point R in $\mathcal{D}$ and α in $\mathcal{J}$, and

$$0 \leq \int_{\alpha_1}^{\alpha_2} \frac{\Delta(x, y, \alpha)}{P^2 + Q^2}\, d\alpha < \pi, \quad \text{where} \quad \Delta(x, y, \alpha) = \begin{vmatrix} P & Q \\ P_\alpha & Q_\alpha \end{vmatrix},$$

and $\alpha_1 < \alpha_2$ are in $\mathcal{J}$, and (x, y) is in $\mathcal{D}$. If (x, y, α_1) is such that the above integral is zero for some $\alpha_2 > \alpha_1$, we will refer to the point (x, y) as an α_1-stationary point, while all other ordinary points of $\mathcal{D}$ will be referred to as α_1-rotation points. In terms of these definitions, we also require that:

(B) for each α in $\mathscr{J}$ there are at most a finite number of α-stationary points on any finite arc of any trajectory of (1.1) corresponding to that α; here, a finite arc of the trajectory of $(x(t, \alpha), y(t, \alpha))$ is the set of points of the (x, y) plane defined for $t_0 \leq t \leq t_1$, t_0 and t_1 being finite but arbitrary.

Condition (A) implies that the vector at each ordinary point of $\mathscr{D}$ either rotates in a clockwise sense as α increases from a point α_1 in $\mathscr{J}$ or is stationary there, depending on whether or not the point is an α_1-rotation point. It is obvious that if the integral in (A) is zero, then the conditions of (A) require that it also vanish if α_2 is replaced by any value between α_1 and α_2. We next observe that if θ is the angle of inclination of the vector at (x, y) for α in $\mathscr{J}$, then $\tan\theta = Q/P$, and since the partial derivative $\theta_\alpha = \Delta(x, y\ \alpha)/(P^2 + Q^2)$, the integral in condition (A) is just $\theta(x, y, \alpha_2) - \theta(x, y, \alpha_1)$.

If we had assumed that P and Q have continuous first derivatives with respect to x and y, then the restrictions in (B) on the set of α-stationary points are unnecessary, since in that case, a somewhat generalized Poincaré-Bendixson theorem due to Urabe and Katsuma [3] may be used in much of what follows. We point out, however, that in the special case considered in Part 3 of this paper, this condition is not satisfied in that the $f(x, y)$ considered there has no partial derivative with respect to y on the line $y = 0$.

For the remainder of the section we assume, for the sake of convenience, that $\mathscr{D}$ is the entire (x, y) plane and that $\mathscr{J}$ is the set of all real numbers. We shall use the following notation:

(a) $\Gamma(R_1, \alpha)$ is the trajectory containing the ordinary point R_1;

(b) if the point R_1 on $\Gamma(R_1, \alpha)$ is reached at $t = t_1$, then the set of points of this trajectory defined for all $t \geq t_1$, the so-called positive half-trajectory of $\Gamma(R_1, \alpha)$ will be denoted by $\Gamma_+(R_1, \alpha)$; the set of points of this trajectory defined for all $t \leq t_1$, the negative half-trajectory, will be denoted by $\Gamma_-(R_1, \alpha)$;

(c) if R_2 is a point on $\Gamma_+(R_1, \alpha)$ corresponding to $t = t_2$, $t_2 > t_1$, then $\Gamma(R_1, R_2, \alpha)$ is the finite arc of this half-trajectory defined by all t in $t_1 \leq t \leq t_2$. We will refer to R_1 and R_2 as the initial point and final points respectively of $\Gamma(R_1, R_2, \alpha)$, and observe that if these coincide, $\Gamma(R_1, R_2, \alpha)$ defines a cycle.

THEOREM 1. Let R_0 be an ordinary point of system (1.1), $\alpha_1 > \alpha_0$, and suppose that the half-trajectories $\Gamma_+(R_0, \alpha_0)$ and $\Gamma_+(R_0, \alpha_1)$ have the point R_1 in

common such that $\Gamma(R_0, R_1, \alpha_0)$ and $\Gamma(R_0, R_1, \alpha_1)$ are finite arcs. Then the region bounded by these finite arcs contains a critical point.

PROOF. We denote by $\mathcal{D}_0$ the region bounded by $\Gamma(R_0, R_1, \alpha_0)$ and $\Gamma(R_0, R_1, \alpha_1)$, and consider an α_0-rotation point P_1 on $\Gamma(R_0, R_1, \alpha_0)$ such that P_1 is distinct from R_0 and R_1. Then $\Gamma(P_1, \alpha_1)$ enters $\mathcal{D}_0$ provided t varies in the appropriate sense; we first assume that $\Gamma(P_1, \alpha_1)$ enters $\mathcal{D}_0$ as t increases, and point out that this assumption implies that for any α_0-rotation point P on $\Gamma(R_0, R_1, \alpha_0)$ and distinct from R_1 and R_0, $\Gamma(P, \alpha_1)$ also enters $\mathcal{D}_0$ as t increases.

If $\Gamma_+(P_1, \alpha_1)$ is contained in $\mathcal{D}_0$, its positive limiting set must either be a cycle, or contain a critical point; in either case $\mathcal{D}_0$ must contain a critical point, and there is nothing to prove.

Hence, assume $\Gamma_+(P_1, \alpha_1)$ intersects the boundary of $\mathcal{D}_0$ at Q_1, the first such point after P_1. The point Q_1 cannot lie on $\Gamma(R_0, R_1, \alpha_1)$ by the uniqueness of solutions of (1.1) for $\alpha = \alpha_1$, and must therefore be an α_0-stationary point on $\Gamma(R_0, R_1, \alpha_0)$, distinct from R_0 and R_1. Now consider an α_0-rotation point P_2 on $\Gamma(P_1, Q_1, \alpha_0)$ and distinct from P_1. Since $\Gamma(P_2, \alpha_1)$ enters $\mathcal{D}_0$ as t increases, then, as before, if $\mathcal{D}_0$ is to contain no critical points, $\Gamma_+(P_2, \alpha_1)$ must intersect $\Gamma(P_1, Q_1, \alpha_0)$ at Q_2, an α_0-stationary point distinct from R_1 and Q_1. We next repeat the same argument in connection with the arc $\Gamma(P_2, Q_2, \alpha_0)$ and conclude that there exists a point Q_3 on it which is an α_0-stationary point distinct from Q_2.

Continuing in the above manner, we conclude the existence of a sequence Q_n of distinct α_0-rotation points on $\Gamma(R_0, R_1, \alpha_0)$, which violates condition (C). Thus the region $\mathcal{D}_0$ must contain a critical point.

In case $\Gamma(P_1, \alpha_1)$ enters $\mathcal{D}_0$ as t decreases, an argument similar to the one given above applies if we consider the negative half-trajectories $\Gamma_-(P_1, \alpha_1)$. This concludes the proof.

In what follows, we assume that all cycles are described in a clockwise sense as t increases.

THEOREM 2. If $\alpha_1 > \alpha_0$, then the cycles $L(\alpha_1)$ and $L(\alpha_0)$ have no points in common.

PROOF. Let P be a point common to $L(\alpha_0)$ and $L(\alpha_1)$. Suppose first that $L(\alpha_0)$ has points exterior to the region bounded by $L(\alpha_1)$. If $\Gamma_+(P, \alpha_0)$ is contained in this region, its positive limiting set is also; but this is impossible, since this set must coincide with $L(\alpha_0)$ which has

points outside this region. Hence, in case $L(\alpha_0)$ has points exterior to the region bounded by $L(\alpha_1)$, $\Gamma_+(P, \alpha_0)$ contains a point Q on $L(\alpha_1)$ and this point must be an α_0-stationary point, and could possibly coincide with P. Let the point Q on $L(\alpha_1)$ correspond to $t = t_0$. Then there exists an $\epsilon > 0$ and sufficiently small so that the point R on $L(\alpha_1)$ corresponding to $t = t_0 + \epsilon$ has the following properties:

(i) R is an α_0-rotation point, and all other points on $L(\alpha_1)$ between Q and R are α_0-rotation points.

(ii) There exists a point R' on $\Gamma_+(Q, \alpha_0)$ such that the line segment RR' is a transversal for the trajectories of (1.1) when $\alpha = \alpha_0$ and $\alpha = \alpha_1$.

(iii) The region bounded by $\Gamma(Q, R', \alpha_0)$, $\Gamma(Q, R, \alpha_1)$, and RR' is free of critical points.

The realization of (i) is an immediate consequence of condition (C). Next, we observe that the tangent vectors to $L(\alpha_0)$ and $L(\alpha_1)$ at Q have the same direction; if not, condition (A) would be violated. From the continuity of $P(x, y, \alpha)$ and $Q(x, y, \alpha)$, it follows that there exists a circle with center at Q inside of which the inclinations of the vectors of each of the fields α_0 and α_1 differ from the inclination of the tangent vector at Q by less than $\pi/8$ radians. Hence, the existence of the transversal described in (ii) and (iii) follows.

Consider now the half-trajectory $\Gamma_-(R, \alpha_0)$. It cannot have points in common with RR' since this line is part of a transversal for $\alpha = \alpha_0$; it cannot have a point in common with $\Gamma(Q, R, \alpha_0)$ by the uniqueness of solutions of (1.1) for $\alpha = \alpha_0$; finally, it cannot intersect $\Gamma(Q, R, \alpha_1)$, since such a point of intersection would necessarily be an α_0-stationary point and would contradict condition (i) above. We conclude that $\Gamma_-(R, \alpha_0)$ is contained in the bounded region described in (iii) above which is free of critical points. This leads to a contradiction; and we conclude that each point of $L(\alpha_0)$ is either on $L(\alpha_1)$ or inside the region bounded by $L(\alpha_1)$. Let Q' be a point common to $L(\alpha_0)$ and $L(\alpha_1)$ and corresponding to $t = t_0'$ on $L(\alpha_1)$. Clearly Q' is an α_0-stationary point, and for $\epsilon > 0$ sufficiently small, the α_0-rotation point S on

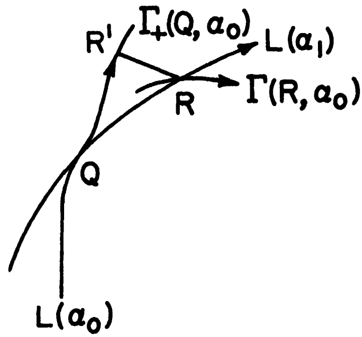

FIGURE 1

$L(\alpha_1)$ corresponding to $t = t_0' - \epsilon$ is so close to Q' so that $\Gamma_+(S, \alpha_0)$ is contained in a bounded region free of critical points; the details of the proof of this follow as in the case above and are omitted. Since we again arrive at a contradiction, we conclude that $L(\alpha_1)$ and $L(\alpha_0)$ have no points in common. This proves the theorem.

THEOREM 3. Let $L(\alpha_0)$ be a cycle of (1.1) stable from outside. Then given any outer neighborhood $\mathcal{N}$ of $L(\alpha_0)$, there exists a α_1 such that for each α in $\alpha_0 < \alpha \leq \alpha_1$, $\mathcal{N}$ contains a cycle $L_1(\alpha)$ stable from inside and a cycle $L_2(\alpha)$ stable from outside. $L_1(\alpha)$ and $L_2(\alpha)$ may coincide.

PROOF. Let $\mathcal{N}_0$ be an outer neighborhood of $L(\alpha_0)$ contained in $\mathcal{N}$ and free of critical points, and let R be an α_0-rotation point in $\mathcal{N}_0$, not on $L(\alpha_0)$, and such that there exists a transversal RR_0 for the α_0-trajectories which connects R to R_0, a point on $L(\alpha_0)$. We assume further that $\Gamma_+(R, \alpha_0)$ intersects RR_0 at R_1, a point between R and R_0, such that $\Gamma(R, R_1, \alpha_0)$ is contained in $\mathcal{N}_0$ and has no points in common with RR_0 except R, R_0, and R_1.

Then from the continuity of the solutions of (1.1) in α, there exists an $\alpha_1 > \alpha_0$ such that for $\alpha_0 < \alpha \leq \alpha_1$:

(i) RR_0 is a transversal for the α-trajectories in the same sense as it is for the α_0-trajectories.

(ii) $\Gamma_+(R, \alpha)$ intersects RR_0 at $R_2 = R_2(\alpha)$, a point between R and R_1; and

(iii) $\Gamma(R, R_2, \alpha)$ is contained in $\mathcal{N}_0$.

Now let S be an α_0-rotation point on $L(\alpha_0)$. Clearly $\Gamma_+(S, \alpha)$ for $\alpha_0 < \alpha \leq \alpha_1$ must be contained in the region bounded by $\Gamma(R, R_2, \alpha)$, $L(\alpha_0)$, and RR_2, for it cannot intersect $L(\alpha_0)$ after S by Theorem 1. It must therefore approach a cycle $L_1(\alpha)$ contained in this region, and $L_1(\alpha)$ must be stable from inside. However, $\Gamma_+(R_2, \alpha)$ also is contained in this region, and must therefore approach a cycle $L_2(\alpha)$ from outside. This completes the proof.

Although system (1.1) may for each α in $\alpha_0 \leq \alpha < \alpha_1$ have a unique cycle stable from outside, it does not necessarily follow that the system will have a cycle for $\alpha = \alpha_1$. It suffices to consider the simple example where $P = y$, $Q = \left[\alpha(x^2 + y^2) + 1\right] y - x$. It is easily seen that for this example a unique cycle defined by $x^2 + y^2 = -1/\alpha$ exists for each $\alpha < 0$, and that this cycle is stable on both sides.

If, however, the region covered by all the cycles for α in $\alpha_0 \leq \alpha < \alpha_1$ can be shown bounded independently of α, then, under some

additional conditions on the location of critical points, the existence of a cycle for $\alpha = \alpha_1$ can be shown. This is done in the next theorem.

THEOREM 4. Suppose that for each α in $\alpha_0 \leq \alpha < \alpha_1$ there exists a cycle $L(\alpha)$ of system (1.1) stable from outside and enclosing any other cycle for that α. Suppose also that every such $L(\alpha)$ is contained in the region bounded by a closed curve C which is independent of α, and that all critical points of (1.1) are either exterior to C or interior to each $L(\alpha)$. Then there exists a cycle of (1.1) for $\alpha = \alpha_1$ stable from inside.

PROOF. Let R_0 be an α_0-rotation point on $L(\alpha_0)$ and consider the half-trajectory $\Gamma_+(R_0, \alpha_1)$; by Theorem 1, it has no points, after the initial point R_0, in common with $L(\alpha_0)$ provided it does not cross the curve C. However, in this case its positive limiting set is contained in the annular region bounded by C and $L(\alpha_0)$, and since this annular region is free of critical points, we conclude that $\Gamma_+(R_0, \alpha_1)$ must approach a cycle $L(\alpha_1)$ and there is nothing more to prove.

Hence, assume that $\Gamma_+(R_0, \alpha_1)$ contains a point R exterior to C. Then for $\epsilon > 0$ sufficiently small, $\Gamma_+(R_0, \alpha_1 - \epsilon)$ also contains a point exterior to C. If $L(\alpha_1 - \epsilon)$ were exterior to $L(\alpha_0)$, it would have to enclose $L(\alpha_0)$, since otherwise there would have to be a critical point in the annular region bounded by C and $L(\alpha_0)$. But if $L(\alpha_1 - \epsilon)$ encloses $L(\alpha_0)$, it must intersect $\Gamma_+(R_0, \alpha_1 - \epsilon)$, which is impossible. Hence, assume that $L(\alpha_1 - \epsilon)$ is interior to $L(\alpha_0)$, and consider $\Gamma_-(R_0, \alpha_1 - \epsilon)$; since we may assume $\alpha_1 - \epsilon > \alpha_0$, this half-trajectory enters the region bounded by $L(\alpha_0)$, and by Theorem 1 it must remain there, since $\Gamma_-(R_0, \alpha_1, - \epsilon)$ cannot intersect $L(\alpha_1 - \epsilon)$. $\Gamma_-(R_0, \alpha_1 - \epsilon)$ must therefore approach a cycle, which cannot be $L(\alpha_1 - \epsilon)$ since $L(\alpha_1 - \epsilon)$ is stable from outside. Hence, there exists another cycle $L_1(\alpha_1 - \epsilon)$ enclosing $L(\alpha_1 - \epsilon)$, which is also contrary to the assumption that $L(\alpha_1 - \epsilon)$ encloses all other cycles for the value of $\alpha = \alpha_1 - \epsilon$. We have a contradiction, and the theorem is proved.

The following example shows that system (1.1) will under the conditions of the last theorem not necessarily have a cycle stable from outside for $\alpha = \alpha_1$. We define $r = (x^2 + y^2)^{\frac{1}{2}}$ and take $P = y$, $Q = - x - f(r)y + \alpha ry$, $\alpha_0 = - 1$, $\alpha_1 = 0$, where $f(r) = r - 1$ for $r < 1$, while $f(r) = 0$ for $r \geq 1$. It is easily seen that for each α in $- 1 \leq \alpha < 0$, system (1.1) will in this case have the unique cycle defined by $r = 1/(1 - \alpha)$ which is stable from outside. However, for $\alpha = 0$,

every circle $r = c$, where $c \geq 1$, is a cycle, and conversely; clearly none of these are asymptotically stable from outside. If we introduce a more general concept of stability, we have, in terms of it, an extension of the last theorem. We say that the cycle L has positive orbital stability, or simply, orbital stability from outside if for every $\epsilon > 0$ there exists a $\delta_\epsilon > 0$ such that every positive half-trajectory originating at a point outside of L and at a distance less than δ_ϵ from L remains at a distance less than ϵ from L.

THEOREM 4a. Under the conditions of the previous theorem there exists a cycle of system (1.1) for $\alpha = \alpha_1$ which is orbitally stable from outside.

PROOF. As before, we denote by $L(\alpha)$ the cycle of (1.1) stable from outside and enclosing all other cycles for that α. We denote by $\mathcal{L}$ the set of points covered by these cycles for α in $\alpha_0 \leq \alpha < \alpha_1$; i.e., $\mathcal{L} = \cup_\alpha L(\alpha)$, α in $\alpha_0 \leq \alpha < \alpha_1$. By Theorem 3, $\mathcal{L}$ is an annular region whose inner boundary is $L(\alpha_0)$. From the proof of Theorem 4, we observe that there must, for $\alpha = \alpha_1$, be a cycle $L(\alpha_1)$ exterior to $L(\alpha_0)$. By Theorem 2, this cycle $L(\alpha_1)$ can have no points in $\mathcal{L}$. Let $\epsilon > 0$ be given, and suppose R is a point exterior to $L(\alpha_1)$ on a α_1-transversal intersecting $L(\alpha_1)$ at S and such that $\Gamma_+(R, \alpha_1)$ intersects this transversal for the first time after R at R_1. We also suppose that the region bounded by $L(\alpha_1)$, $\Gamma(R, R_1, \alpha_1)$ and RR_1 is free of critical points, and is within a distance of ϵ of $L(\alpha_1)$. Suppose first that R_1 is either identical with R or between R and S; it clearly follows that in this case $L(\alpha_1)$ is orbitally stable from outside, and there is nothing more to prove.

Hence suppose R is between R_1 and S on the transversal. Then for $\epsilon_0 > 0$ and sufficiently small, $\Gamma_+(R, \alpha_1 - \epsilon_0)$ intersects the transversal at R_0 such that R is again between R_0 and S. Consider the negative half-trajectory $\Gamma_-(R, \alpha_1 - \epsilon_0)$; since R is exterior to $L(\alpha_1 - \epsilon_0)$, the limiting set of this half-trajectory must either be $L(\alpha_1 - \epsilon_0)$, or a cycle $L_1(\alpha_1 - \epsilon_0)$ exterior to $L(\alpha_1 - \epsilon_0)$. But each possibility contradicts the definition of $L(\alpha_1 - \epsilon_0)$, since the first suggests that this cycle is unstable from outside, and the second, that $L(\alpha_1 - \epsilon_0)$ does not enclose all other cycles for $\alpha = \alpha_1 - \epsilon_0$.

This completes the proof of the theorem.

2. AN EQUATION FOR RELAXATION OSCILLATIONS

We now consider the system

$$\dot{x} = y, \quad \dot{y} = -g(x) - f(x, y)y \tag{2.1}$$

where $f(x, y)$ and $g(x)$ satisfy Lipschitz conditions in every bounded domain of the phase plane. We also assume that:

(i) $xg(x) > 0$ for $x \neq 0$, and $G(x) \longrightarrow \infty$ as $|x| \longrightarrow \infty$ where $G(x) = \int_0^x g(s)\,ds$;

(ii) $f(0, 0) < 0$, $f(x, y) > -M$ for some constant M;

(iii) there exists a constant $\mu > 0$ and function $h(y)$ continuous on $-\infty < y < +\infty$ such that $f(x, y) > \mu G(x) - h(y)$ for all x and y;

(iv) there exist numbers $a < 0$ and $b > 0$, and continuous solutions $y_0(x)$ and $y_1(x)$ of the equation $f(x, y)y + g(x) = 0$ such that for $x \leq a$, $y_0(x) > 0$ and strictly increasing, while for $x \geq b$, $y_1(x) < 0$ and strictly increasing, and such that for these values of x, there is no solution y of this equation such that $|y| < |y_i(x)|$, $i = 0, 1$;

(v) there exists a point (x_1, y_1) on the line $y = y_0(a) + M_1(x - a)$ such that for $x \geq x_1$ all points (if any) on the graph of $f(x, y)y + g(x) = 0$ which are above the x-axis lie above this line; there exists a point (x_2, y_2) on $y = y_1(b) + M_2(x - b)$ such that for $x \leq x_2$ all points (if any) on the graph of this equation and below the x-axis lie below this line; here

$$M_1 = M + \max_{a \leq x \leq 0} [-g(x)/y_0(a)] ,$$

$$M_2 = M + \max_{0 \leq x \leq b} [-g(x)/y_1(b)] .$$

REMARK. The following set of conditions imply the conditions (iv) and (v) above:

(iv') there exist positive numbers a and b such that for (x, y) in $R_1 \cup R_2$, where R_1 is defined by $x < -a$, $0 < y < b$, and R_2 by $x > a$, $-b < y < 0$,

1) $$f_x,\ f_y,\ g'(x) \text{ exist;}$$

2) $$b\left\{\min_{|y| \leq b} (yf_y + f)\right\} > |g| \quad ;$$

3) $$-\frac{gf_x}{f} + g' < 0 \quad ;$$

(v') if we assume, without loss of generality, that $h(y) \geq 0$ for all y, then there exists a positive number c such that for $|y| > c$, $yx(y) \leq y^2[\max(M_1, M_2) + \epsilon]^{-1}$; here $x(y)$ is the unique solution of $\mu G(x) = h(y)$ for which $yx(y) \geq 0$ for all $y \neq 0$, M_1 and M_2 are as defined in (v) above, and $\epsilon > 0$ is an arbitrary constant.

We first show that (iv') implies (iv). Suppose (x, y) is in R_1, then clearly $F(x, 0) = g(x) < 0$, where $yf(x, y) + g(x) = F(x, y)$. Also for fixed x, $F(x, b) - F(x, a) = bF_y(x, \xi)$, $0 < \xi < b$. Hence $F(x, b) = g(x) + b(\xi f_y(x, \xi) + f(x, \xi)) > g(x) + |g(x)| = 0$ and we conclude that there exists $y_0 = y_0(x)$, $0 < y_0 < b$, such that $y_0 f(x, y_0) + g(x) = 0$. By 2), $F_y > 0$; hence $y_0(x)$ is unique in R_1, has a derivative given by

$$y_0'(x) = -\frac{yf_x + g'}{yf_y + f} \ ;$$

and, by 3), $y_0'(x) > 0$.

By applying a similar argument to R_2, a solution $y_1(x)$ of $F(x, y) = 0$ is obtained in this region for which $y_1'(x) > 0$. Hence we have condition (iv).

To show that (v') implies (v), we supposed that $y > 0$, and have, by virtue of (iii), that $F(x, y) > y(\mu G(x) - h(y)) + g(x)$, where $F(x, y)$ is defined as above. Then if $x \geq x(y)$ and $y > 0$, we have $F(x, y) \geq g(x) > 0$. However, since $y \geq \bar{M}x(y)$ whenever $y > c$, where $\bar{M} = \max(M_1, M_2) + \epsilon$, we see that for $y > c$ the curve $x = x(y)$ must lie above the line $y = \bar{M}x$, and must therefore intersect the line $y - b = M_1(x + a)$ at some $x = c_1 > c$ and remain above it for $x > c_1$.

A similar argument shows that the region $y < 0$, $x \leq x(y)$ can contain no solutions of $F(x, y) = 0$, and that its right boundary, the curve $x = x(y)$ for $y < 0$, intersects the line $y + b = M_2(x - a)$ at some $x = -c_2 < -c$, and remains below it for $x < -c_2$. Thus we see that (v') implies (v).

We now show that the special case

$$f(x, y) = \frac{x^4}{|y| + 1} - 1,$$

$g(x) = \frac{x}{2}$ satisfies conditions (i) - (iii), (iv') and (v'). Conditions (i) and (ii) are obviously satisfied. For condition (iii), we take $\mu = 1$, and $h(y) = 2|y| + 2$; we must then have $x^4 - (|y| + 1) > (|y| + 1)x^2 - 2(|y| + 1)^2$. Regarded as a quadratic inequality in x^2 for each fixed y,

this will be satisfied provided $(|y| + 1)^2 < 4[2(|y| + 1)^2 - (|y| + 1)]$ for all y, which is certainly the case. We find that

$$yf_y + f = \frac{x^4}{(|y|+1)^2} - 1 \quad ;$$

if $a = 2$, $b = 1$, we have clearly that

$$\frac{x^4}{(|y|+1)^2} - 1 > \frac{x^4}{4} - 1 > \frac{|x|}{2} \quad \text{for} \quad |x| > 2 \quad ;$$

moreover,

$$\frac{x^4}{(|y|+1)^2} - 1 > 1 \quad \text{for} \quad |x| > 2 \quad .$$

We also see that for $|x| > 2$, $|y| < 1$,

$$yf_x + g' = \frac{4x^3y}{|y| + 1} + \frac{1}{2} < -2 + \frac{1}{2} < 0 \quad \text{for} \quad y = -g/f \quad .$$

Hence all parts of (iv') are satisfied.

Next, the equation $\mu G(x) = h(y)$, or in this case, $x^2 = 2|y| + 2$, has the solution $x(y) = \pm (2|y| + 2)^{\frac{1}{2}}$ where we take the $+$ sign for $y > 0$, and the $-$ sign for $y < 0$. Clearly, in this case, corresponding to any positive constant k there exists a number $c = c(k)$ such that $yx(y) \leq ky^2$ for each y such that $|y| > c$. We thus see that condition (v') is satisfied.

We now state and prove the following theorem.

THEOREM 5. Under conditions (i) - (v) listed above, there exists a cycle of system (2.1) stable from inside, and a cycle orbitally stable from outside (in the sense of the definition immediately preceding the statement of Theorem 4a). These cycles need not necessarily be distinct.

PROOF. Consider the system

$$\dot{x} = y, \quad \dot{y} = -g(x) - F(x, y, \alpha)y \quad , \tag{2.2}$$

where $F(x, y, \alpha) = -\alpha(x^2 + y^2) + f(x, y)y$. It is easily seen that for this system, condition (A) of Part 2 is satisfied. In fact, the set of

α-stationary points is the x-axis with the origin, the only critical point of (2.2) as well as (2.1), removed. Suppose that the finite arc defined in $c \leq t \leq d$ of some trajectory intersects the x-axis for $t = t_i$, $i = 1, 2, \ldots$ such that the points $x(t_i)$ have a limit point $x_o \neq 0$ corresponding to $t = t_o$. We may clearly assume that $t_i \longrightarrow t_o$. By definition, $y(t_i) = 0$, $i = 1, 2, \ldots$; hence there exists a sequence $\xi_i \longrightarrow t_o$ such that $\dot{y}(\xi_i) = 0$, $i = 1, 2, \ldots$. Suppose for some i, $\dot{x}(\xi_i) = 0$; then the arc clearly contains a critical point, the origin, which is impossible. Hence for each i, $\dot{x}(\xi_i) \neq 0$. But this, together with $\dot{y}(\xi_i) = 0$, implies that the slope of the arc at the point corresponding to $t = \xi_i$ must be zero. Since this slope is continuous in t, it follows that the slope of the arc at $(x_o, 0)$ is zero. This is impossible, and we conclude that any finite arc of an arbitrary trajectory of (2.2) can intersect the x-axis at no more than a finite number of points. Condition (B), then, is also satisfied, and we may use the results of the previous section.

We first observe that for each $\alpha < 0$, system (2.2) has a cycle stable from outside. For in this case, we may find a constant c_1 large enough so that $F(x, y, \alpha) > 0$ on the closed curve $y^2 - 2G(x) = c_1$ and this implies that this closed curve is an outer Bendixson curve, in the sense that trajectories which meet it, pass inside it as t increases; cf. Levinson and Smith, [2]. Similarly for $c_2 > 0$ sufficiently small, $F(x, y, \alpha) < 0$ on the closed curve $y^2 + 2G(x) = c_2$; i.e., this curve is an inner Bendixson curve. Hence for each $\alpha < 0$ the existence of a cycle stable from outside follows.

Our theorem will clearly follow from Theorems 4 and 4a if we show that there exists a closed curve bounding the region covered by the cycles $L(\alpha)$ of (2.2) for $\alpha_o \leq \alpha < 0$, for any fixed $\alpha_o < 0$. To this end, we first show that if

$$y_m = \max_{(x,y) \text{ on } L(\alpha)} |y| \quad ,$$

then y_m is bounded independently of α. We will show this only for $y > 0$; the proof for $y < 0$ is entirely similar.

Consider any half-trajectory $\Gamma_+(R)$ of system (2.1) where R is a point $(x_o, 0)$ such that $x_o \leq a$. Define $\Gamma(R, S)$ to be $\Gamma_+(R)$ in case $\Gamma_+(R)$ has no points other than R in common with the x-axis; if $\Gamma_+(R)$ has the point S in common with the x-axis, we define $\Gamma(R, S)$ to be the finite arc of $\Gamma_+(R)$ from R to S. We may clearly assume that R and S are the only points common to $\Gamma(R, S)$ and the x-axis in this last case. We show now that all positive ordinates of $\Gamma(R, S)$ are less than y_1 where (x_1, y_1) is the point defined in condition (v). We

observe first that for $x_0 \leq x \leq a$, the arc $\Gamma(R, S)$ must have positive slope wherever it intersects the curve $y = y_0(x)$; but this is impossible since on $y = y_0(x)$ any trajectory of (2.1) has zero slope. We thus conclude that $\Gamma(R, S)$ must be bounded above by $y = y_0(x)$ on $x_0 \leq x \leq a$.

Now consider the equation $y = \bar{y}(x)$ of $\Gamma(R, S)$ for $x > a$. Clearly $\bar{y}'(x) = -g(x)/\bar{y}(x) - f(x, \bar{y}(x))$. If $y = \bar{y}(x)$ intersects the line $y = y_0(a) + M_1(x - a)$ at $x = \bar{x}_1$, then we must have $\bar{y}'(\bar{x}_1) \geq M_1$; but we also have

$$\bar{y}'(\bar{x}_1) < \max_{a \leq x \leq 0} - \left[g(x)/y_0(a)\right] + M \leq M_1 \quad ,$$

which leads to a contradiction. Hence $\Gamma(R, S)$ cannot pass above this line for $x > a$.

We now show that the slope of $\Gamma(R, S)$ at (x_1, y_1) is negative. If not, it must be positive, since the point (x_1, y_1) cannot lie on the curve $f(x, y)y + g(x) = 0$; cf. condition (v). Now from conditions (i) and (iv) there exists a positive $\bar{x}_2 > x_1$ such that $f(\bar{x}_2, y_1) > 0$. Hence $f(\bar{x}_2, y_1)y_1 + g(\bar{x}_2) > 0$, and we conclude that the slope of the trajectory at $(\bar{x}_2, y_1)$ is negative. But this implies that the slope of the trajectory at any point $(\bar{x}_2, y)$ below the line $y = y_0(a) + M_1(x - a)$ and above the x-axis must be negative; for if not, there exists a point $(\bar{x}_2, \bar{y}_2)$ below the line and such that $\bar{y}_2 > y_1$, at which the slope of trajectory is zero, which in turn implies that $f(\bar{x}_2, \bar{y}_2)\bar{y}_2 + g(\bar{x}_2) = 0$. This is impossible by condition (v) and we conclude that the slope of $\Gamma(R, S)$ at $x = \bar{x}_2$, if defined there, must be negative. Since we had assumed the slope of $\Gamma(R, S)$ to be positive at $x = x_1$, there must be a value of x in $x_1 < x < \bar{x}_2$ at which $\Gamma(R, S)$ has zero slope, again a contradiction with condition (v). It follows that the maximum ordinate on $\Gamma(R, S)$ is bounded by y_1.

We now turn to system (2.2) and show that if $\alpha < 0$, then $y(\alpha) < y_1$, where $y(\alpha)$ is the greatest ordinate on the cycle $L(\alpha)$ enclosing all others for that value of α. Suppose the cycle $L(\alpha)$ crosses the negative x-axis at $x_0(\alpha)$ and consider the trajectory of system (2.1) containing the point $(x_0(\alpha) + a, 0)$. For $y > 0$, $L(\alpha)$ must clearly remain under this trajectory, since all points for which $y > 0$ are α-rotation points for any α, and if $L(\alpha)$ were to have a point in common with this trajectory, its slope at that point must be less than that of the trajectory there. However, the ordinates of the trajectory are bounded by y_1, and we therefore conclude that the ordinates of $L(\alpha)$ are also bounded by y_1.

As was previously asserted, the details of the proof that y_2 of condition (v) is a lower bound of the ordinate of $L(\alpha)$ will be omitted,

as they follow along the same lines as above.

We now show that a bound on $|y|$ for (x, y) on $L(\alpha)$ implies that $L(\alpha)$ must be contained in a region bounded independently of α for $\alpha_0 \leq \alpha < 0$. If $x = x(t, \alpha)$, $y = y(t, \alpha)$ define $L(\alpha)$ parametrically, then these functions satisfy

$$\begin{aligned}\frac{d}{dt}\left(y^2/2 + G(x)\right) &= \left(\alpha r^2 - f(x, y)\right)y^2 \\ &\leq -\Big(f(x, y) - \mu G(x) + \mu y^2/2 + \mu G(x) \\ &\qquad - \mu y^2/2\Big)y^2 \\ &\leq -\mu y^2\left(y^2/2 + G(x)\right) + \left(h(y) + \mu y^2/2\right)y^2\end{aligned}$$

Hence,

$$\frac{d}{dt}\left(y^2/2 + G(x)\right) + \mu y^2\left(y^2/2 + G(x)\right) \leq y^2 H$$

where

$$\bar{y} = \max_{i=1,2} (|y_i|) \quad \text{and} \quad H = \max_{|y| \leq \bar{y}} \left(h(y) + \mu y^2/2\right) .$$

Multiplying this last inequality by

$$E_0(t, \alpha) = \exp \mu \int_{t_0}^{t} y^2(s, \alpha)ds \quad ,$$

where t_0 is arbitrary, and integrating from t_0 to $t_0 + T$ where T is the period of the functions $x(t, \alpha)$ and $y(t, \alpha)$, we obtain

$$\begin{aligned}&\left[y^2(t_0, \alpha)/2 + G\left(x(t_0, \alpha)\right)\right]\left(E_0(t_0 + T, \alpha) - 1\right) \\ &\qquad \leq H\left(E_0(t_0 + T, \alpha) - 1\right)/\mu \quad ;\end{aligned}$$

and since $E_0(t_0 + T, \alpha) > 1$, we obtain $y^2(t_0, \alpha)/2 + G(x(t_0, \alpha)) < H/\mu$. Since H is independent of α, and t_0 is arbitrary, it follows that every point on $L(\alpha)$ is within the region bounded by the curve $y^2 + 2G(x) = 3H/\mu$. This completes the proof of the theorem.

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VIII. A SURVEY OF LYAPUNOV'S SECOND METHOD

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Lyapunov's second method, as it was called by Lyapunov himself in his dissertation [13], has in recent years become an important tool in the stability theory of solutions of ordinary differential equations. This paper is a survey of its development to date. More specifically, the present paper brings together in one place, and relates to one another, criteria for various types of stability that are based upon considerations of certain gauge functions, which have come to be known in the literature as Lyapunov functions. All results are formulated in n-dimensional Euclidean space although most hold, with minor modifications, in general Banach spaces. For the sake of completeness, the majority of theorems is stated with complete proofs.

No attempt has been made to include in this paper the large number of available results concerning stability in the first approximation as well as criteria for instability. A treatment of these and related questions is contained in [21]. A discussion of open problems in the area of stability theory may be found in [5], [28].

The paper is divided into four parts: I. Definitions; II. Stability; III. Asymptotic Stability; IV. Boundedness and Asymptotic Stability in the Large.

I. DEFINITIONS

1. Throughout the sequel the basic differential equation will be

$$\dot{x} = f(t, x)$$

where $f(t, x)$ is a function with values in Euclidean n-space R^n which is defined and continuous on some set $I \times S = \{(t, x) \in R \times R^n \mid t \geq T \geq 0, \|x\| < r\}$. It is assumed that $f(t, x)$ is sufficiently smooth on $I \times S$ such that, given any $(t_0, x_0) \in I \times S$, there exists for all $t \geq t_0$ a unique solution in S, $x = F(t, t_0, x_0)$, which depends continuously upon

(t_0, x_0) and equals x_0 at t_0. Moreover, $x = 0$ will always be supposed to be a (trivial) solution so that $f(t, 0) \equiv 0$ on I.

DEFINITION 1. The solution $x = 0$ is said to be

(1.1) stable [13]: if given any $\epsilon > 0$ and any $t_0 \in I$ there exists a $\delta(\epsilon, t_0) > 0$ such that $\|x_0\| < \delta$ implies $\|F(t, t_0, x_0)\| < \epsilon$ for $t \geq t_0$.

(1.2) uniformly stable [35]: if given any $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that $t_0 \in I$, $\|x_0\| < \delta$ imply $\|F(t, t_0, x_0)\| < \epsilon$ for $t \geq t_0$.

(1.3) quasi-asymptotically stable: if given any $t_0 \in I$ there exists a $\delta(t_0) > 0$ such that $\|x_0\| < \delta$ implies $F(t, t_0, x_0) \longrightarrow 0$ as $t \longrightarrow \infty$.

(1.4) asymptotically stable [13]: if it is both stable and quasi-asymptotically stable.

(1.5) quasi-equiasymptotically stable: if given any $t_0 \in I$ there exists a $\delta(t_0) > 0$ such that $\|x_0\| < \delta$ implies $F(t, t_0, x_0) \longrightarrow 0$ as $t \longrightarrow \infty$ uniformly on $\|x_0\| < \delta$.

(1.6) equiasymptotically stable [24]: if it is both stable and quasi-equiasymptotically stable.

(1.7) quasi-uniform-asymptotically stable [25]: if there exists a $\delta_0 > 0$ such that $t_0 \in I$, $\|x_0\| < \delta_0$ imply $F(t, t_0, x_0) \longrightarrow 0$ as $t \longrightarrow \infty$ uniformly on $t_0 \in I$, $\|x_0\| < \delta_0$.

(1.8) uniform-asymptotically stable [35]: if it is both uniformly stable and quasi-uniform-asymptotically stable.

(1.9) exponential-asymptotically stable [15]: if there exists a $\lambda > 0$ and, given any $\epsilon > 0$, a $\delta(\epsilon) > 0$ such that $t_0 \in I$, $\|x_0\| < \delta$ imply $\|F(t, t_0, x_0)\| < \epsilon \exp[-\lambda(t - t_0)]$ for $t \geq t_0$.

It is clear that exponential-asymptotic stability implies uniform-asymptotic stability which, in turn, implies all other types of stability. Furthermore, the following implications, among others, hold:

$$(1.7) \Longrightarrow (1.5) \Longrightarrow (1.3),$$
$$(1.6) \Longrightarrow (1.4) \Longrightarrow (1.1),$$
$$(1.2) \Longrightarrow (1.1) \quad .$$

If $f(t, x)$ is lipschitzian on some set $I \times H$, $H \subset S$, for a function $k(t) \geq 0$, which is defined and piecewise continuous on I, (1.5) implies (1.6).

If $f(t, x)$ is lipschitzian on some set $I \times H$, $H \subset S$, for a constant $k > 0$, (1.7) implies (1.8), [25].

If $f(t, x)$ is independent of t or periodic in t on $I \times S$, (1.1) implies (1.2) and (1.4) implies (1.8), (cf., e.g., [25]).

If $f(t, x)$ is linear in x on $I \times S$, (1.4) implies (1.6) and (1.8) implies (1.9), [24], [25].

Finally, if $f(t, x)$ is scalar, (1.4) implies (1.6), [24].

For examples illustrating some of these relations among the various types of stability, we refer to [25].

2. Let $V(t, x)$ be a real scalar function, defined and locally lipschitzian on some set $I_o \times S_o = \{(t, x) \in R \times R^n \mid t \geq T_o \geq 0, \|x\| < r_o\}$, and such that, given any $x \in S_o$, $V(t, x)$ is continuous on I_o, and $V(t, 0) \equiv 0$ on I_o.

In what follows, it will be assumed throughout that $S_o \cap S$ contains a neighborhood N of $x = 0$, say $N = \{x \in S_o \cap S \mid \|x\| < \rho\}$, where $\rho > 0$ is some fixed constant, and that $I_o \cap I = I = [T, \infty)$, $T \geq 0$.

$V(t, x)$ is said to be positive definite on $I \times N$ if given any ϵ, $0 < \epsilon < \rho$, there exists a $\mu(\epsilon) > 0$ such that $V(t, x) \geq \mu$ for $t \in I$, $\epsilon \leq \|x\| < \rho$. Similarly, $V(t, x)$ is said to be negative definite on $I \times N$ if $-V(t, x)$ is positive definite on $I \times N$.

Corresponding to $V(t, x)$ we define on $I \times N$ the function $V'(t, x)$,

$$V'(t, x) = \overline{\lim_{h \to 0+}} \; (1/h) \{V(t + h, x + hf(t, x)) - V(t, x)\},$$

which is said to be the generalized (upper right-hand) total derivative of $V(t, x)$ by virtue of the equation $\dot{x} = f(t, x)$, [39].

LEMMA 1. If $(t_o, x_o) \in I \times N$, the upper right-hand derivate of $V(t, F(t, t_o, x_o))$ at $t = t_o$ equals $V'(t_o, x_o)$.

The proof is trivial and therefore is omitted.

DEFINITION 2. A real scalar function $V(t, x)$ is said to be a Lyapunov function on $I \times N$ for the

equation $\dot{x} = f(t, x)$ if it is defined, locally lipschitzian and positive definite on $I \times N$; if, given any $x \in N$, $V(t, x)$ is continuous on I and $V(t, 0) \equiv 0$ on I; and if $V'(t, x) \leq 0$ on $I \times N$.

This terminology differs from that commonly used in the Russian literature. There a real scalar function $V(t, x)$ is said to be a Lyapunov function on $I \times N$ if it is defined, of class C^1, and positive definite on $I \times N$, and if $V'(t, x)$ is negative definite on $I \times N$. Malkin and Moissejev [26] appear to have been the first to consider Lyapunov functions under less stringent regularity assumptions.

The vector function $f(t, x)$ is said to be on $I \times N$ of class C^1 with respect to x if, on $I \times N$, $f(t, x)$ is continuous and has continuous first partial derivatives with respect to the components of x.

LEMMA 2. If, on $I \times N$, $f(t, x)$ is of class C^1 with respect to x, there exists a topological mapping $s = \emptyset(t)$ of I to $I^* = [T^*, \infty)$, $T^* \geq 0$, which transforms $dx/dt = f(t, x)$ into $dx/ds = f^*(s, x)$ such that $|\partial f^*_i(s, x)/\partial x_j| \leq 1$ on $I^* \times N$, $1 \leq i, j \leq n$. Moreover, if $V(s, x)$ is a Lyapunov function on $I^* \times N$ for the latter equation such that $V'(s, x)$ is negative definite on $I^* \times N$, $W(t, x) = V(\emptyset(t), x)$ is a Lyapunov function on $I \times N$ for the former such that $W'(t, x)$ is negative definite on $I \times N$.

PROOF. Let $\varphi(t)$ be a real scalar function, defined and continuous on $t \geq 0$ such that, for any $x \in N$, $|\partial f_i(t, x)/\partial x_j| \leq \varphi(t)$ on I, $1 \leq i, j \leq n$; we may assume $\varphi(t) \geq 1$ on I and $\varphi(t) \longrightarrow \infty$ with t. Define $\emptyset(t) = \int_0^t \varphi(u)du$ and let $\Psi(s)$ be its inverse function. The substitution $s = \emptyset(t)$ transforms the equation $dx/dt = f(t, x)$ into

$$\frac{dx}{ds} = \frac{f(\Psi(s), x)}{\varphi(\Psi(s))} = f^*(s, x)$$

where $f^*(s, x)$ is defined and of class C^1 with respect to x on the set $I^* \times N$, $I^* = [\emptyset(T), \infty)$. Clearly, $|\partial f^*_i(s, x)/\partial x_j| \leq 1$ on $I^* \times N$, $1 \leq i, j \leq n$.

If $V(s, x)$ is a Lyapunov function on $I^* \times N$ for $dx/ds = f^*(s, x)$ such that $V'(s, x)$ is negative definite on $I^* \times N$, $W(t, x) = V(\emptyset(t), x)$ has obviously the same regularity properties on $I \times N$ as $V(s, x)$ has on $I^* \times N$; moreover, on $I \times N$, $W(t, x)$ is positive definite and $W'(t, x) = \varphi(t)V'(\emptyset(t), x)$. Thus, $W(t, x)$ is a Lyapunov

function on $I \times N$ for $dx/dt = f(t, x)$ such that $W'(t, x)$ is negative definite on $I \times N$.

For the sake of completeness, we state without proof the following lemma due in its original formulation to Massera [24], which will be needed later.

LEMMA 3. Given any real scalar function $g(t)$, defined and positive on every compact interval $J \subset L = [0, \infty)$, $g(t) \longrightarrow 0$ as $t \longrightarrow \infty$, and any real scalar function $h(t)$, defined, of class C, positive, and non-decreasing on L, there exists for any integer $k > 0$ a real scalar function $G(t)$, defined, of class C^k, and increasing together with its first k derivatives on L, $G^{(i)}(0) = 0$, $0 \leq i \leq k$, such that for any real scalar function $g^*(t)$, defined and satisfying, on L, $0 \leq g^*(t) \leq cg(t)$ for some constant $c > 0$, the integrals

$$\int_0^\infty G^{(i)}(g^*(t))h(t)dt, \qquad 0 \leq i \leq k \quad ,$$

converge uniformly in g^*.

II. STABILITY

1. In 1892 Lyapunov [13] proved the following theorem.

THEOREM 1. If there exists, on $I \times N$, a Lyapunov function $V(t, x)$, $x = 0$ is stable.

PROOF. Given any ϵ, $0 < \epsilon < \rho$, there is a $\mu(\epsilon) > 0$ such that $V(t, x) \geq \mu$ for $t \in I$, $\epsilon \leq \|x\| < \rho$ and, given any $t_o \in I$, there is a $\delta(t_o, \epsilon) > 0$ such that $V(t_o, x) < \mu$ on $\|x\| < \delta$. If $\|x_o\| < \delta$, then $\|F(t, t_o, x_o)\| < \epsilon$ for $t \geq t_o$; for, otherwise, $\|F(t, t_o, x_o)\| = \epsilon$ at some time $t = t' > t_o$ so that $\mu \leq V(t', t_o, x_o)) \leq V(t_o, x_o) < \mu$, which is absurd.

Lyapunov's original formulation requires $V(t, x)$ to be of class C^1 on $I \times N$. At the expense of restricting $f(t, x)$ to be, on $I \times S$, of class C^1 with respect to x, Persidskii [33], [34] obtained the following converse theorem.

THEOREM 2. If $f(t, x)$ is, for some integer $k \geq 1$,

of class C^k with respect to x on $I \times S$ and, for every $x \in S$, of class C^{k-1} on I and such that, given any $(t_o, x_o) \in I \times S$, $F(t, t_o, x_o)$ is continuable to the whole of I, and if $x = 0$ is stable, there exists on some set $J \times N = \{(t, x) \mid t \geq T' \geq T, \|x\| < \rho\}$, $J \times N \subset I \times S$, a Lyapunov function $V(t, x)$ of class C^k such that, for any compact interval $K \subset J$, $V'(t, x)$ is negative definite on $K \times N$.

PROOF. By hypothesis, given any $(t_o, x_o) \in I \times S$, $F(t, t_o, x_o)$ exists for all $t \in I$ and is in S; hence for any fixed $T' \in I$, $F(T', t, x)$ is of class C^k on $I \times S$. Moreover, the i-th component, F_i, of $F(T', t, x)$ satisfies $\partial F_i/\partial t + \operatorname{grad} F_i \cdot f(t, x) = 0$ on $I \times S$.

Let $\rho > 0$ be so chosen that given any ϵ, $0 < \epsilon < \rho$, and any $t_o \in I$, there exists a $\delta(t_o, \epsilon) > 0$ such that $\|x_o\| < \delta$ implies $\|F(t, t_o, x_o)\| < \epsilon$ for $t \geq t_o$, and let $J \times N = \{(t, x) \mid t \geq T', \|x\| < \rho\}$. Then given any ϵ, $0 < \epsilon < \rho$, there is a $\delta^*(\epsilon) = \delta(T', \epsilon) > 0$ such that $\|F(T', t, x)\| \geq \delta^*$ for $t \geq T'$, $\epsilon \leq \|x\| < \rho$.

Let P be a constant positive definite matrix and let $Q(t)$ be a matrix, defined, of class C^k on J, positive definite on every compact interval $K \subset J$, and such that $\int^\infty \|Q(t)\| dt < \infty$. Put $M(t) = P + \int_t^\infty Q(u)du$ and define

$$V(t, x) = (F(T', t, x), \quad M(t)F(T', t, x)) .$$

Clearly, $V(t, x)$ is of class C^k on $J \times N$; moreover, on $J \times N$, $V(t, x)$ is positive definite, and

$$V'(t, x) = - (F(T', t, x), \quad Q(t)F(T', t, x)) .$$

COROLLARY. If $f(t, x)$ is linear in x on $I \times S$ and if $x = 0$ is stable, there exists a real quadratic form in x, $V(t, x)$, with coefficients that are functions of t of class C^1 on I such that $V(t, x)$ is a Lyapunov function on $I \times S$ and such that, for every compact interval $J \subset I$, $V'(t, x)$ is negative definite on $J \times S$.

Note that if $f(t, x)$ is linear in x on $I \times S$, then, given any $(t_o, x_o) \in I \times S$, $F(t, t_o, x_o) = X(t)X^{-1}(t_o)x_o$ for all $t \in I$ where $X(t)$ is a non-singular (matrix) solution of the associated matrix equation.

2. Persidskii [35], (cf., also, [27]), was the first to give

sufficient conditions for uniform stability.

THEOREM 3. If there exists, on $I \times N$, a Lyapunov function $V(t, x)$ such that $V(t, x) \longrightarrow 0$ with x uniformly on I, $x = 0$ is uniformly stable.

The proof differs very little from that of Theorem 1 and therefore is omitted.

The following converse theorem is due to Kurzweil [12].

THEOREM 4. If, on $I \times S$, $f(t, x)$ is of class C^1 with respect to x and such that, given any $(t_0, x_0) \in I \times S$, $F(t, t_0, x_0)$ is continuable to the whole of I, and if $x = 0$ is uniformly stable, there exists on some set $J \times N = \{(t, x) \mid t \geq T' \geq T, \|x\| < \rho\}$, $J \times N \subset I \times S$, a Lyapunov function $V(t, x)$ of class C^1 such that $V(t, x) \longrightarrow 0$ with x uniformly on J.

For the proof we refer to [12].

A result similar to Theorem 4 was proved by Krasovskii [11] under stronger regularity assumptions on $f(t, x)$; his proof is based upon Barbasin's method of sections [2].

The next theorem due to Massera [25] generalizes an analogous result of Lyapunov [13] concerning the uniform asymptotic stability of $x = 0$ when $f(t, x)$ is linear in x and independent of t on $I \times S$.

THEOREM 5. If, on $I \times S$, $f(t, x)$ is linear in x and independent of t and if $x = 0$ is stable, hence uniformly stable, then given any even integer $m > 0$ there exists a real algebraic form $V(x)$ of degree m which is a Lyapunov function on $I \times S$.

For the proof we refer to [25].

III. ASYMPTOTIC STABILITY

1. We begin with the following theorem.

THEOREM 6. If there exists, on $I \times N$, a Lyapunov function $V(t, x)$ such that $V'(t, x)$ is negative definite on $I \times N$, $x = 0$ is stable; moreover, given

any $t_0 \in I$ and any ρ', $0 < \rho' < \rho$, there exists a $\delta(t_0, \rho') > 0$ and, given any x_0 in $\|x\| < \delta$ and any ϵ, $0 < \epsilon < \rho'$, there exists a $\tau_0(t_0, \rho', \epsilon) > 0$ and a $t_1(t_0, x_0) \in [t_0, t_0 + \tau_0)$ such that $\|F(t_1, t_0, x_0)\| < \epsilon$.

PROOF. By Theorem 1, $x = 0$ is stable. Thus, given any ρ', $0 < \rho' < \rho$, and any $t_0 \in I$, there is a $\delta(t_0, \rho') > 0$ such that $\|x_0\| < \delta$ implies $\|F(t, t_0, x_0)\| < \rho'$ for $t \geq t_0$. Given any ϵ, $0 < \epsilon < \rho'$, there are constants $\mu(\epsilon) > 0$, $\nu(\epsilon) > 0$ such that $V(t, x) \geq \mu$, $V'(t, x) \leq -\nu$ for $t \in I$, $\epsilon \leq \|x\| < \rho'$. Let $\lambda(t_0, \rho') = \sup\{V(t_0, x) \mid \|x\| < \delta\}$ and put $\tau_0(t_0, \rho', \epsilon) = \lambda/\nu$.

Given any x_0 in $\|x\| < \delta$, either $\|x_0\| \geq \epsilon$ or not. In the first case, $\|F(t, t_0, x_0)\| \geq \epsilon$ for some $t \geq t_0$. If $\epsilon \leq \|F(t, t_0, x_0)\| < \rho'$ throughout $[t_0, t_0 + \tau_0]$ then $V'(t, x) \leq -\nu$ on $[t_0, t_0 + \tau_0]$ whence

$$\mu \leq V\Big(t_0 + \tau_0, F(t_0 + \tau_0, t_0, x_0)\Big) \leq V(t_0, x_0) - \nu\tau_0 \leq 0 \quad ,$$

which is absurd. Hence there is a $t_1(t_0, x_0) \in (t_0, t_0 + \tau_0)$ such that $\|F(t_1, t_0, x_0)\| < \epsilon$. If $\|x_0\| < \epsilon$, put $t_1 = t_0$. Note that in either case $V(t_1, F(t_1, t_0, x_0)) \leq V(t_0, x_0)$.

COROLLARY 1. If the hypotheses of Theorem 6 hold, given any $t_0 \in I$ and any ρ', $0 < \rho' < \rho$, there exists a $\delta(t_0, \rho') > 0$ and, given any x_0 in $\|x\| < \delta$ and any null sequence $\{\epsilon_n\}$, $0 < \epsilon_n < \rho'$, there exist a divergent non-decreasing sequence $\{t_n\}$, $t_n \geq t_0$, and a divergent sequence $\{t'_n\}$, $t'_n(t_0, \rho', \epsilon_1, \ldots, \epsilon_n) > 0$, such that $t_n < t_0 + t'_n$ and $\|F(t_n, t_0, x_0)\| < \epsilon_n$, $n = 1, 2, \ldots$.

PROOF. Given any ρ', $0 < \rho' < \rho$, and any $t_0 \in I$, let $\delta(t_0, \rho') > 0$ and $\lambda(t_0, \rho') > 0$ be as in the proof of Theorem 6. Given any null sequence $\{\epsilon_n\}$, $0 < \epsilon_n < \rho'$, there are sequences $\{\mu_n\}$, $\mu_n(\epsilon_n) > 0$, and $\{\nu_n\}$, $\nu_n(\epsilon_n) > 0$, such that $V(t, x) \geq \mu_n$, $V'(t, x) \leq -\nu_n$ for $t \in I$, $\epsilon_n \leq \|x\| < \rho'$. Put $\tau_{n-1}(t_0, \rho', \epsilon_n) = \lambda/\nu_n$. It follows from the proof of Theorem 6 that there is a sequence $\{t_n\}$, $t_n \geq t_{n-1}$, such that $\|F(t_n, t_0, x_0)\| < \epsilon_n$, $n = 1, 2, \ldots$. Clearly, $t_0 \leq t_n < t_0 + \Sigma_0^{n-1}\tau_k$; let $t'_n(t_0, \rho', \epsilon_1, \ldots, \epsilon_n) = \Sigma_0^{n-1}\tau_k$. Since $\{t_n\}$ is non-decreasing it either diverges or converges to some t_∞, $T < t_\infty < \infty$. In the latter case, the hypothesis that $F(t, t_\infty, 0) \equiv 0$ be unique would be violated; hence $\{t_n\}$ diverges, and so does obviously $\{t'_n\}$.

COROLLARY 2. If the hypotheses of Theorem 6 hold and if $V(t, x)$ is bounded on $I \times N$, given any $t_0 \in I$ and any ρ', $0 < \rho' < \rho$, there exists a $\delta(t_0, \rho') > 0$ and, given any x_0 in $\|x\| < \delta$ and any ϵ, $0 < \epsilon < \rho'$, there exists a $\tau(\rho', \epsilon) > 0$, independent of t_0, and a $t_1(t_0, x_0) \in [t_0, t_0 + \tau)$ such that $\|F(t_1, t_0, x_0)\| < \epsilon$.

For the proof replace, in the proof of Theorem 6, $\lambda(t_0, \rho')$ by an upper bound of $V(t, x)$ on $I \times N$.

The assumptions of Theorem 6 are not sufficient for the asymptotic stability of $x = 0$, even when $f(t, x)$ is restricted to be linear in x on $I \times N$, [24], nor are they necessary. Indeed, asymptotic stability of $x = 0$ does not even imply the weaker requirement that there exist, on $I \times N$, a Lyapunov function $V(t, x)$ and a real scalar positive definite function $U(t, x)$ such that $V'(t, x) + U(t, x) \longrightarrow 0$ as $t \longrightarrow \infty$ uniformly on $\rho_1 \leq \|x\| \leq \rho_2$, for every positive $\rho_1, \rho_2 < \rho$. This can be seen from a slight modification of an example in [24].

2. THEOREM 7. If $f(t, x)$ is bounded on $I \times N$ and if there exists, on $I \times N$, a Lyapunov function $V(t, x)$ such that $V'(t, x)$ is negative definite on $I \times N$, $x = 0$ is asymptotically stable.

PROOF. By Theorem 1, $x = 0$ is stable. Thus, given any ρ', $0 < \rho' < \rho$, and any $t_0 \in I$, there exists a $\delta(t_0, \rho') > 0$ such that $\|x_0\| < \delta$ implies $\|F(t, t_0, x_0)\| < \rho'$ for $t \geq t_0$.

If $x = 0$ were not asymptotically stable, there would exist a $\tau_0 \in I$ and a ξ in $\|x\| < \delta(\tau_0)$ such that for some ϵ, $0 < \epsilon < \rho'$, and some divergent sequence $\{\tau_n\}, \{\tau_n\} \in I$, we would have $\|F(\tau_n, \tau_0, \xi)\| = \epsilon$. Let $M > 0$ be an upper bound of $f(t, x)$ on $I \times N$. Then $\|F(t, \tau_0, \xi) - F(\tau_n, \tau_0, \xi)\| \leq M\,|t - \tau_n|$ for $t \geq \tau_0$, $n = 1, 2, \ldots$, whence $\|F(t, \tau_0, \xi)\| \geq \epsilon/2$ on $J_n = [\tau_n - \epsilon/2M, \tau_n + \epsilon/2M]$; clearly, we may assume $J_{n+1} \cap J_n = 0$, $n = 1, 2, \ldots$, and $\tau_1 > \tau_0 + \epsilon/2M$. There exist constants $\mu(\epsilon) > 0$, $\nu(\epsilon) > 0$ such that $V(t, x) \geq \mu$, $V'(t, x) \leq -\nu$, for $t \geq T$, $\epsilon/2 \leq \|x\| < \rho'$. Hence

$$\mu \leq V\Big(\tau_n + \epsilon/2M,\ F(\tau_n + \epsilon/2M,\ \tau_0,\ \xi)\Big) \leq V(\tau_0, \xi) - \nu n\epsilon/M \quad ,$$

which is absurd for sufficiently large n.

COROLLARY. If $f(t, x)$ is independent of t or periodic in t on $I \times N$, and if there exists, on $I \times N$, a Lyapunov function $V(t, x)$ such that

$V'(t, x)$ is negative definite on $I \times N$, $x = 0$ is uniform-asymptotically stable.

The assumptions of Theorem 7, in general, do not even imply the equiasymptotic stability of $x = 0$, [23], [24].

Theorem 7, which is due to Marachkov [23], generalizes Lyapunov's classical theorem [13] on asymptotic stability. Aside from more stringent regularity assumptions on the function $V(t, x)$, Lyapunov assumes the hypotheses of Theorem 6 and the further condition that $V(t, x) \longrightarrow 0$ with x uniformly on I. These assumptions where shown by Persidskii [35] to be sufficient (cf. Theorem 13) and by Malkin [22] to be necessary (cf. Theorem 15) for the uniform asymptotic stability of $x = 0$. Zubov, [42], [43], [44], proved, among others, criteria weaker than Lyapunov's theorem which are sufficient as well as necessary for the asymptotic stability for $x = 0$. These criteria are based on considerations of functions which are closely related to Lyapunov functions.

The requirement that $V(t, x) \longrightarrow 0$ with x uniformly on I is too restrictive to be implied by the equiasymptotic stability, even when the hypotheses on $V'(t, x)$ are weakened similarly as mentioned above, [24]. Yet, as Theorem 8, [24], shows, whenever $f(t, x)$ is linear in x on $I \times N$, the existence of a Lyapunov function satisfying a weaker additional requirement does follow from the asymptotic stability of $x = 0$, which in this case is equivalent with equiasymptotic stability.

THEOREM 8. If $f(t, x)$ is linear in x on $I \times N$ and, for every $x \in N$, of class C^{k-1} on I for some integer $k \geq 1$, and if $x = 0$ is asymptotically, hence equiasymptotically, stable, there exists a Lyapunov function $V(t, x)$, defined and of class C^k on $I \times N$, such that $V'(t, x)$ is negative definite on $I \times N$. Moreover, $V(t, x)$ has the property that, given any ξ, $0 < \xi < \rho$, and any $\sigma \geq T$, there are constants $\eta(\xi)$, $0 < \eta \leq \xi$, and $\tau(\sigma, \xi) \geq \sigma$ such that, for any $s \in [T, \sigma]$ and any y in $\|y\| < \eta$, the inequalities $t \geq \tau$, $V(t, x) \leq V(s, y)$ imply $\|x\| < \xi$.

PROOF. Since $f(t, x)$ is linear in x on $I \times N$, given any $(t_o, x_o) \in I \times N$, we have $F(t, t_o, x_o) = X(t)X^{-1}(t_o)x_o$ where $X(t)$ is a non-singular (matrix) solution of the associated matrix equation. Thus, for $t \in I$,

$$\|x\| = \|X(t)X^{-1}(T)F(T, t, x)\| \leq \|X(t)X^{-1}(T)\| \; \|F(T, t, x)\| \quad .$$

Let $g(t) = \|X(t)X^{-1}(T)\|$; by hypothesis, $g(t)$ is defined on I and $g(t) \longrightarrow 0$ as $t \longrightarrow \infty$. It follows that, given any $\epsilon > 0$, there exists a $\mu(\epsilon) > 0$ such that $\|F(T, t, x)\| \geq \mu$ for $t \geq T$, $\|x\| \geq \epsilon$; moreover, $\|F(T, t, x)\| \longrightarrow \infty$ with t uniformly in x on $\|x\| \geq \xi$ for every $\xi > 0$.

Let $G(t)$ be the function associated with $g(t)$ and $h(t) \equiv 1$ by Lemma 3, let $\phi(t, x) = \|F(T, t, x)\|$, and define

$$V(t, x) = \int_T^\infty G\Big(g(s)\phi(t, x)\Big)ds + \int_t^\infty G\Big(g(s)\phi(t, x)\Big)ds \quad .$$

According to Lemma 3, $V(t, x)$ is defined and of class C^k on $I \times N$. If $t \geq T$, $\|x\| \geq \epsilon$, then $\phi(t, x) \geq \mu$ and hence

$$V(t, x) \geq \int_T^\infty G\Big(g(s)\phi(t, x)\Big)ds \geq \int_T^\infty G\Big(g(s)\mu\Big)ds$$

so that $V(t, x)$ is positive definite on $I \times N$. Given any $(t_0, x_0) \in I \times N$,

$$V\Big(t, F(t, t_0, x_0)\Big) = \int_T^\infty G\Big(g(s)\phi(t_0, x_0)\Big)ds + \int_t^\infty G\Big(g(s)\phi(t_0, x_0)\Big) \quad .$$

whence

$$\dot{V}\Big(t, F(t, t_0, x_0)\Big) = -G\Big(g(t)\phi(t_0, x_0)\Big) \quad .$$

Thus, on $I \times N$, $V'(t, x) = -G(g(t)\phi(t, x)) \leq -G(\|x\|)$ so that $V'(t, x)$ is negative definite on $I \times N$.

Given any ξ, $0 < \xi < \rho$, and any $\sigma \geq T$, let $\eta = \xi$ and choose $\tau(\sigma, \xi) \geq \sigma$ so large that if $u \in [T, \sigma]$ and y in $\|y\| < \xi$, then $t \geq \tau$ and

$$2 \int_T^\infty G\Big(g(s)\phi(u, y)\Big)ds \geq \int_T^\infty G\Big(g(s)\phi(t, x)\Big)ds$$

imply $\|x\| < \xi$. Clearly, this is always possible. Hence, if $u \in [T, \sigma]$ and y in $\|y\| < \xi$; then the inequalities $t \geq \tau$, $V(t, x) \leq V(u, y)$ imply $\|x\| < \xi$, for

$$\int_T^\infty G\Big(g(s)\phi(t, x)\Big)ds \leq V(t, x) \leq V(u, y) \leq 2 \int_T^\infty G\Big(g(s)\phi(u, y)\Big)ds \quad .$$

The next theorem due to Massera [25] shows that the hypothesis on $f(t, x)$ in Theorem 7 can be omitted at the expense of restricting $V'(t, x)$ further.

THEOREM 9. If there exists, on $I \times N$, a real scalar function $V(t, x)$, locally lipschitzian and positive definite, and if there exists a real scalar function $W(t)$, defined, continuous and increasing for $t \geq 0$, $W(0) = 0$, such that $V'(t, x) \leq - W(V(t, x))$ on $I \times N$, $x = 0$ is asymptotically stable.

PROOF. By Theorem 1, $x = 0$ is stable. Hence, given any ρ', $0 < \rho' < \rho$, and any $t_0 \in I$, there exists a $\delta(t_0, \rho') > 0$ such that $\|x_0\| < \delta$ implies $\|F(t, t_0, x_0)\| < \rho'$ for $t \geq t_0$.

Since $V'(t, F(t, t_0, x_0)) \leq - W(V(t, F(t, t_0, x_0)))$ for $t \geq t_0$,

$$\int_{V(t_0,x_0)}^{V} \frac{dV}{W(V)} \leq - (t - t_0)$$

and hence $V(t, F(t, t_0, x_0)) \longrightarrow 0$ as $t \longrightarrow \infty$. Thus, $F(t, t_0, x_0) \longrightarrow 0$ as $t \longrightarrow \infty$.

3. The next two theorems concern sufficient conditions for the equiasymptotic stability of $x = 0$. Theorem 10 is due to Massera [24]. Theorem 11 was originally stated by Malkin [18] who merely proved stability; in this formulation, Theorem 11 follows as a corollary from theorems on stability given earlier by Halikoff [8], [9]. Massera [24] extended Malkin's proof to yield equiasymptotic stability.

THEOREM 10. If there exists, on $I \times N$, a Lyapunov function $V(t, x)$ such that $V'(t, x)$ is negative definite on $I \times N$, and if $V(t, x)$ has the further property that, given any ξ, $0 < \xi < \rho$, and any $\sigma \geq T$, there are constants $\eta(\xi)$, $0 < \eta \leq \xi$, and $\tau(\sigma, \xi) \geq \sigma$ such that, for any $s \in [T, \sigma]$ and any y in $\|y\| < \eta$, the inequalities $t \geq \tau$, $V(t, x) \leq V(s, y)$ imply $\|x\| < \xi$, then $x = 0$ is equiasymptotically stable.

PROOF. By Theorem 6, $x = 0$ is stable and, given any ρ', $0 < \rho' < \rho$, and any $t_0 \in I$, there exists a $\delta(t_0, \rho') > 0$ and, given any x_0 in $\|x\| < \delta$ and any η, $0 < \eta < \rho'$, there exist constants $\sigma(t_0, \eta) > 0$ and $t' \in [t_0, t_0 + \sigma)$ such that $\|F(t', t_0, x_0)\| < \eta$. Let

ϵ be given, $0 < \epsilon < \rho'$, and let $\eta = \eta(\epsilon) \leq \epsilon$, $\tau(t_o, \epsilon) \geq t_o + \sigma$ be the constants corresponding to ϵ and $t_o + \sigma$ according to the hypothesis. Then $V(t, F(t, t_o, x_o)) \leq V(t', F(t', t_o, x_o))$ for $t \geq t'$ and, a fortiori, for $t \geq \tau$ whence $\|F(t, t_o, x_o)\| < \epsilon$ for $t \geq \tau$.

Theorem 8 states that, if $f(t, x)$ is linear in x on $I \times N$, the assumptions of Theorem 10 are also necessary for the equiasymptotic stability of $x = 0$.

Theorem 11 below shows that the hypothesis on $V'(t, x)$ in Theorem 10 can be weakened at the expense of requiring that $V(t, x) \longrightarrow 0$ with x uniformly on I.

> THEOREM 11. If there exist, on $I \times N$, real scalar positive definite functions $U(t, x)$, $V(t, x)$ such that $V(t, x)$ is continuous on $I \times N$, $V(t, x) \longrightarrow 0$ with x uniformly on I and $V'(t, x) + U(t, x) \longrightarrow 0$ as $t \longrightarrow \infty$ uniformly on $\rho_1 \leq \|x\| \leq \rho_2$ for every positive $\rho_1, \rho_2 < \rho$, $x = 0$ is equiasymptotically stable.

PROOF. Given any non-increasing null sequence $\{\gamma_n\}$, $0 < \gamma_n < \rho$, let $\mu_n(\gamma_n) = \inf \{V(t, x) \mid t \geq T, \|x\| = \gamma_n\}$ and $\lambda_n(\gamma_n) = \sup \{V(t, x) \mid t \geq T, \|x\| < \gamma_n\}$; clearly, $\mu_n > 0$, $0 \leq \lambda_n < \infty$, and $V(t, x) = \mu_n$ implies $t \geq T$, $\beta_n \leq \|x\| \leq \gamma_n$ for some $\beta_n(\gamma_n) > 0$. There exist constants $\alpha_n(\gamma_n) > 0$, $\nu_n(\gamma_n) > 0$ such that $V(t, x) < \mu_n$ for $t \geq T$, $\|x\| < \alpha_n$ and $U(t, x) \geq 2\nu_n$ for $t \geq T$, $\alpha_n \leq \|x\| < \rho$; clearly, $\alpha_n < \beta_n < \gamma_n$. Moreover, there exists a divergent sequence $\{\tau_n\}$, $\tau_n(\gamma_n) > T$, such that $V'(t, x) + U(t, x) < \nu_n$ for $t \geq \tau_n$, $\alpha_n \leq \|x\| \leq \gamma_1$, i.e., $V'(t, x) < -\nu_n$ for $t \geq \tau_n$, $\alpha_n \leq \|x\| \leq \gamma_1$; without loss of generality we may assume $\tau_{n+2} > \tau_{n+1} + \lambda_n/\nu_{n+1}$, $n = 1, 2, \ldots$.

By the hypothesis concerning the continuous dependence of $F(t, t_o, x_o)$ upon (t_o, x_o), given any $t_o \geq T$, there exists a $\delta(\gamma_1, t_o) = \delta^*(\tau_1, t_o) > 0$ such that $\|x_o\| < \delta$ implies $\|F(t, t_o, x_o)\| < \alpha_1$ on $[t_o, t_o + \tau_1]$. Then $\|F(t, t_o, x_o)\| < \gamma_1$ for $t \geq t_o + \tau_1$; for, otherwise, $\|F(t, t_o, x_o)\| = \gamma_1$ at some time $t = t' > t_o + \tau_1$ which implies $V(t', F(t', t_o, x_o)) \geq \mu_1$. Since by construction $V(t_o + \tau_1, F(t_o + \tau_1, t_o, x_o)) < \mu_1$, there exists a $t'' \in (t_o + \tau_1, t']$ such that $V(t'', F(t'', t_o, x_o)) = \mu_1$ whence $\beta_1 \leq \|F(t'', t_o, x_o)\| \leq \gamma_1$ and $V'(t'', F(t'', t_o, x_o)) < -\nu_1$, which is absurd. This proves the stability of $x = 0$.

Now, either $\|F(t_o + \tau_2, t_o, x_o)\| < \alpha_2$ or not. In the first case, we can conclude by an argument similar to the one above that $\|F(t, t_o, x_o)\| < \gamma_2$ for $t \geq t_o + \tau_2$. If, on the other hand,

$\|F(t_0 + \tau_2, t_0, x_0)\| \geq \alpha_2$, there exists a $t_2 \in J_2 = [t_0 + \tau_2, t_0 + \tau_2 + \lambda_1/\nu_2]$ such that $\|F(t_2, t_0, x_0)\| < \alpha_2$; for, otherwise, $\alpha_2 \leq \|F(t, t_0, x_0)\| \leq \gamma_1$ throughout J_2 whence $V'(t, F(t, t_0, x_0)) < -\nu_2$ throughout J_2 and $V(t_0+\tau_2+\lambda_1/\nu_2, F(t_0+\tau_2+\lambda_1/\nu_2, t_0, x_0)) < V(t_0+\tau_2, F(t_0+\tau_2, t_0, x_0)) - \lambda_1 \leq 0$, which is absurd. Again, by an argument similar to the one above, $\|F(t, t_0, x_0)\| < \gamma_2$ for $t \geq t_2$ and hence, a fortiori, for $t \geq t_0 + \tau_2 + \lambda_1/\nu_2$. Thus, in any case, $\|F(t, t_0, x_0)\| < \gamma_2$ for $t \geq t_0 + \tau_3$. Continuing this way, we conclude that $\|F(t, t_0, x_0)\| < \gamma_n$ for $t \geq t_0 + \tau_{n+1}$, $n = 1, 2, \ldots$.

REMARK. Note that $\{\tau_n\}$ is independent of t_0; it depends solely upon $\{\gamma_n\}$ and the functions $U(t, x)$, $V(t, x)$.

COROLLARY. If $f(t, x)$ is lipschitzian for some constant $k > 0$ on $I \times N$, and if the hypotheses of Theorem 11 are satisfied, $x = 0$ is uniform-asymptotically stable.

The proof parallels that of Theorem 11. Observe that by the hypothesis on $f(t, x)$, given any $(t_0, x_0) \in I \times N$, $\|F(t, t_0, x_0)\| \leq \|x_0\| \exp k(t - t_0)$ for $t \geq t_0$.

In view of the corollary, the assumptions of Theorem 11 are not necessary for the equiasymptotic stability of $x = 0$, even when $f(t, x)$ is linear in x on $I \times N$.

The next theorem may, in certain cases, be easier to apply than the foregoing Theorems 10 and 11.

THEOREM 12. If $x = 0$ is uniformly stable and if there exists, on $I \times N$, a Lyapunov function $V(t, x)$ such that $V'(t, x)$ is negative definite on $I \times N$, $x = 0$ is equiasymptotically stable.

PROOF. Clearly, the quasi-equiasymptotic stability of $x = 0$ needs only to be proved.

Let a positive constant $\rho' < \rho$ be so chosen that given any ϵ, $0 < \epsilon \leq \rho'$, there exists a $\delta(\epsilon) > 0$ such that $t_0 \in I$, $\|x_0\| < \delta$ imply $\|F(t, t_0, x_0)\| < \epsilon$ for $t \geq t_0$, and let $\delta_0 = \delta(\rho')$. By Theorem 6, given any $t_0 \in I$ and any x_0 in $\|x\| < \delta_0$, there exists a $\tau(t_0, \epsilon) > 0$ and a $t' \in [t_0, t_0 + \tau]$ such that $\|F(t', t_0, x_0)\| < \delta(\epsilon)$. Hence $\|F(t, t_0, x_0)\| < \epsilon$ for $t \geq t'$ and, a fortiori, for $t \geq t_0 + \tau$.

4. The theorems that follow concern the uniform-asymptotic stability of $x = 0$. Theorem 13 is Lyapunov's classical theorem on asymptotic

stability, [13], as formulated by Persidskii [27], [31], [32], [35].

THEOREM 13. If there exists, on $I \times N$, a Lyapunov function $V(t, x)$ such that $V(t, x) \longrightarrow 0$ with x uniformly on I and $V'(t, x)$ is negative definite on $I \times N$, $x = 0$ is uniform-asymptotically stable.

PROOF. By Theorem 3, $x = 0$ is uniformly stable. Thus, given any ρ', $0 < \rho' < \rho$, there exists a $\delta(\rho') = \delta_0 > 0$ such that $t_0 \in I$, $\|x_0\| < \delta_0$ imply $\|F(t, t_0, x_0)\| < \rho'$ for $t \geq t_0$. Given any ϵ, $0 < \epsilon < \rho'$, there exists a $\delta(\epsilon) > 0$ and, by Corollary 2 to Theorem 6, given any x_0 in $\|x\| < \delta_0$ there exist constants $\tau(\rho', \delta(\epsilon)) = \tau^*(\epsilon) > 0$ and $t' \in [t_0, t_0 + \tau^*)$ such that $\|F(t', t_0, x_0)\| < \delta(\epsilon)$. Therefore, $\|F(t, t_0, x_0)\| < \epsilon$ for $t \geq t'$ and, a fortiori, for $t \geq t_0 + \tau^*$.

The severity of the assumptions in Theorem 13 is illustrated by the following theorem.

THEOREM 14. If $V(t, x)$ is a Lyapunov function on $I \times N$ such that $V(t, x) \longrightarrow 0$ with x uniformly on I and $V'(t, x)$ is negative definite on $I \times N$, then given any constants ρ', ρ'', $0 < \rho'' < \rho' < \rho$, there exist constants $\lambda(\rho', \rho'') > 0$, $\nu(\rho'') > 0$ such that $W(t, x) = \exp(\lambda t)V(t, x)$ satisfies $W'(t, x) \leq -\nu$ for $t \geq T$, $\rho'' \leq \|x\| < \rho'$.

The proof is trivial and therefore is omitted.

The next theorem due to Malkin [22] is a partial converse to Theorem 13. In Malkin's original formulation the first order partial derivatives of $f(t, x)$ with respect to x are required to be bounded on $I \times N$.

THEOREM 15. If $f(t, x)$ is, for some integer $k \geq 1$, of class C^k with respect to x on $I \times N$ and, for every $x \in N$, of class C^{k-1} on I, and if $x = 0$ is uniform-asymptotically stable, there exists on some set $I \times M$, $M \subset N$, a Lyapunov function $V(t, x)$ of class C^k such that $V(t, x) \longrightarrow 0$ with x uniformly on I and $V'(t, x)$ is negative definite on $I \times M$.

PROOF. By Lemma 2, we may assume without loss of generality that $|\partial f_i(t, x)/\partial x_j| \leq 1$ on $I \times N$, $1 \leq i, j \leq n$.

Let a positive constant $\rho^* < \rho$ be so chosen that, given any ϵ,

$0 < \epsilon \leq \rho^*$, there exists a $\delta(\epsilon) > 0$ such that $t \in I$, $\|x\| < \delta$ imply $\|F(t + s, t, x)\| < \epsilon$ for $s \geq 0$. By hypothesis, there exists a $\delta_0 > 0$ and, given any $\eta > 0$, however small, a $\tau(\eta) \geq T$ such that $t \in I$, $\|x\| < \delta_0$ imply $\|F(t + s, t, x)\| < \eta$ for $s \geq \tau$. Let $\rho' = \min(\delta_0, \delta(\rho^*))$ and let M be the open sphere $\|x\| < \rho'$. Clearly, for any $s \geq 0$, $F(t + s, t, x)$ is of class C^k on $I \times M$.

The boundedness, on $I \times M$, of the first partial derivatives of $f(t, x)$ with respect to x implies that, for any $s \geq 0$, the first partial derivatives of $\|F(t + s, t, x)\|$ with respect to t and x are bounded, on $I \times M$, by $h(s) = k\exp(\lambda s)$ where $k = k(\rho^*) > 0$ and $\lambda > 0$ are certain constants independent of (t, x). More specifically, k is an upper bound of $\|f(t, x)\|$ for $t \in I$, $\|x\| < \rho^*$.

Given any non-increasing null sequence $\{\gamma_n\}$, $0 < \gamma_n < \rho'$, there exists an increasing divergent sequence $\{t_n\}$, $t_n(\gamma_n) > 0$, such that $(t, x) \in I \times M$ implies $\|F(t + s, t, x)\| < \gamma_n$ for $s \geq t_n$. Let $g(s)$ be a real scalar function, defined and continuous, positive, non-increasing for $s \geq 0$, $g(s) \longrightarrow 0$ as $s \longrightarrow \infty$, such that given any $(t, x) \in I \times M$, $\|F(t + s, t, x)\| \leq g(s)$ on $[0, t_2]$ and $g(t_{n+1}) = \gamma_n$, $n = 1, 2, \ldots$. Then $g(t_{n+1}) \leq g(s) \leq g(t_n)$ on $[t_n, t_{n+1}]$ and hence $\|F(t + s, t, x)\| < \gamma_n \leq g(s)$ on $[t_{n+1}, t_{n+2}]$, $n = 1, 2, \ldots$; therefore $\|F(t + s, t, x)\| \leq g(s)$ for $s \geq 0$.

Let $G(s)$ be the function associated by Lemma 3 with the functions $g(s)$, $h(s)$ and define

$$V(t, x) = \int_0^\infty G(\|F(t + s, t, x)\|)ds \quad .$$

Clearly, $V(t, x)$ is defined on $I \times M$ and, by Lemma 3, is of class C^k. Moreover, the first partial derivatives of $V(t, x)$ with respect to x are bounded on $I \times M$ so that $V(t, x) \longrightarrow 0$ with x uniformly on I.

Given any $(t, x) \in I \times M$, we have for $s \geq 0$

$$\|F(t + s, t, x) - x\| \leq \int_t^{t+s} \|f(u, F(u, t, x))\| \, du < ks$$

whence, for $s \in [0, \|x\|/2k]$, $\|F(t + s, t, x)\| > (1/2)\|x\|$. Therefore,

$$V(t, x) > \int_0^{\|x\|/2k} G(\|F(t + s, t, x)\|)ds > \|x\|/(2k)G(\|x\|/2)$$

so that $V(t, x)$ is positive definite on $I \times M$.

Clearly, given any $(t_0, x_0) \in I \times M$,

$$V\Big(t, F(t, t_0, x_0)\Big) = \int_0^\infty G\Big(\|F(t + s, t_0, x_0)\|\Big)ds$$

whence

$$\dot{V}\Big(t, F(t, t_0, x_0)\Big) = - G\Big(\|F(t, t_0, x_0)\|\Big)$$

so that $V'(t, x)$ is negative definite on $I \times M$.

If $f(t, x)$ is linear in x on $I \times N$, we have the following important result of Malkin [20] (cf., also, [1]), which he obtained prior to Theorem 15.

THEOREM 16. If $f(t, x)$ is linear in x and bounded on $I \times N$, and if $x = 0$ is uniform-asymptotically stable, hence exponential-asymptotically stable, given any real scalar function $W(t, x)$, defined and continuous on $I \times N$, which is a positive definite form in x of degree $m > 0$, there exists, on $I \times N$, a real scalar function $V(t, x)$ of class C^1 which is a positive definite form in x of degree m such that $V(t, x) \longrightarrow 0$ with x uniformly on I and $V'(t, x) = - W(t, x)$ on $I \times N$.

For the proof we refer to [20], and to [1] if $W(t, x)$ is a quadratic form in x independent of t.

Theorem 16 generalizes an analogous result of Lyapunov [13] for linear systems with constant coefficients. It also extends an earlier theorem of Malkin [16] on the necessity of the existence of a Lyapunov function of class C^1 on $I \times N$, satisfying the hypotheses of Theorem 13, in order that, given any $(t_0, x_0) \in I \times N$, every solution $F(t, t_0, x_0)$ satisfy Persidskii's condition [31]: $\|F(t, t_0, x_0)\| < M\|x_0\|$ for $t \geq t_0 + \tau(M, x_0)$ where $\tau > 0$ and M is any given constant, $0 < M < 1$. For the relation of these results to work by Perron [29], [30] on the boundedness of every solution of linear non-homogeneous systems with bounded forcing term, we refer to [1], [16], [19].

The following theorem of Massera [25] proves a stronger contention than Theorem 15 under considerably weaker hypotheses.

THEOREM 17. If $f(t, x)$ is locally lipschitzian on $I \times N$ and if $x = 0$ is uniform-asymptotically stable, there exists on some set $I \times M$, $M \subset N$, a Lyapunov function $V(t, x)$, possessing partial derivatives with respect to t and x of any order, such that $V(t, x) \longrightarrow 0$ with x uniformly on I and $V'(t, x)$ is negative definite on $I \times M$. If $f(t, x)$ is lipschitzian on $I \times N$, the partial derivatives of $V(t, x)$ are bounded on $I \times M$; if, on $I \times N$, $f(t, x)$ is independent of t or periodic in t, $V(t, x)$ is independent of t or periodic in t on $I \times M$.

For the proof we refer to [25].

Theorem 17 contains earlier results of Massera [25] and Barbasin [2] on the existence of a Lyapunov function $V(t, x)$ on $I \times M$, $M \subset N$, such that $V'(t, x)$ is negative definite on $I \times M$ for the case of asymptotic (and hence uniform-asymptotic) stability of $x = 0$ when $f(t, x)$ is independent of t or periodic in t on $I \times N$. These results are converse theorems to the corollary to Theorem 7.

The next theorems, though essentially elementary, may be useful in some applications. Theorem 18 is related to Theorem 12 on the equi-asymptotic stability of $x = 0$; it generalizes an analogous theorem of Massera [25] based on the assumptions of Theorem 9. If $f(t, x)$ is locally lipschitzian on $I \times N$, the converse to Theorem 18 is obviously true as a consequence of Theorem 17. Theorems 19 and 20 due to Massera [25] show that the hypotheses of Theorem 9 can be implemented so as to yield the uniform-asymptotic stability of $x = 0$.

THEOREM 18. If $x = 0$ is uniformly stable and if there exists, on $I \times N$, a bounded Lyapunov function $V(t, x)$ such that $V'(t, x)$ is negative definite on $I \times N$, $x = 0$ is uniform-asymptotically stable.

The proof is very similar to that of Theorem 12 and therefore is omitted.

THEOREM 19. If $f(t, x)$, is lipschitzian on $I \times N$ and if there exists a real scalar function $V(t, x)$ defined, locally lipschitzian and positive definite on $I \times N$, and a real scalar function $W(t)$, defined, continuous and increasing for $t \geq 0$, $W(0) = 0$, such that $V'(t, x) \leq - W(V(t, x))$ on $I \times N$, then $x = 0$ is uniform-asymptotically stable.

PROOF. By Theorem 18, the uniform stability of $x = 0$ needs only to be proved.

Let $f(t, x)$ be lipschitzian for the constant $k > 0$ on $I \times N$, and let $K > 0$ be an upper bound of $V(t, x)$ on $I \times N$. Given any ϵ, $0 < \epsilon < \rho$, there is a $\mu(\epsilon) > 0$ such that $V(t, x) \geq \mu$ for $t \in I$, $\epsilon \leq \|x\| < \rho$. Let $\tau(\epsilon) = \int_{\mu(\epsilon)}^{K} dV/W(V)$ and $\delta(\epsilon) = \epsilon \exp(-k\tau)$. Then $t_0 \in I$, $\|x_0\| < \delta$ imply $\|F(t, t_0, x_0)\| < \epsilon$ for $t \in [t_0, t_0 + \tau]$; for $t > t_0 + \tau$, it follows that

$$\int_{V(t_0,x_0)}^{V} \frac{dV}{W(V)} < \int_{K}^{\mu} \frac{dV}{W(V)}$$

and thus $V(t, F(t, t_0, x_0)) < \mu$, which implies $\|F(t, t_0, x_0)\| < \epsilon$.

THEOREM 20. If there exist a real scalar function $V(t, x)$, defined, locally lipschitzian and positive definite on $I \times N$, and real scalar functions $W(t)$, $g(t)$, defined and continuous on I, $W(t)$ increasing for $t \geq 0$, $W(0) = 0$, $g(t)$ positive on I and $\int_0 dt/g(t) = \infty$, such that $V'(t, x) \leq -W(V(t, x))$ and $(x, f(t, x)) \leq \|x\| g(\|x\|)$ on $I \times N$, then $x = 0$ is uniform-asymptotically stable.

PROOF. By Theorem 18, the uniform stability of $x = 0$ needs only to be proved.

Given any ϵ, $0 < \epsilon < \rho$, let $\mu(\epsilon) > 0$, $\tau(\epsilon) > 0$ be as in the proof of Theorem 19, and choose $\delta(\epsilon) > 0$ such that $\int_\delta^\epsilon dt/g(t) > \tau(\epsilon)$. Since

$$\frac{d\|x\|^2}{dt} = 2(x, f(t, x)) \leq 2\|x\| g(\|x\|)$$

on $I \times N$, given any $t_0 \in I$ and any x_0 in $\|x\| < \delta$, it follows that for $t \in [t_0, t_0 + \tau]$

$$\int_{\|x_0\|}^{\|x\|} \frac{d\|x\|}{g(\|x\|)} \leq \int_{\delta}^{\epsilon} \frac{d\|x\|}{g(\|x\|)}$$

whence $\|F(t, t_0, x_0)\| < \epsilon$; and, by the same argument as in the proof of Theorem 19, $\|F(t, t_0, x_0)\| < \epsilon$ for $t \geq t_0 + \tau$.

IV. BOUNDEDNESS AND ASYMPTOTIC STABILITY IN THE LARGE

1. We shall now suppose that the basic differential equation

$$\dot{x} = f(t, x)$$

is defined on $I \times R^n$ and that $f(t, x)$ has the same regularity properties on $I \times R^n$ as it was assumed in (I.) to have on $I \times S$, except that for boundedness results we shall not require $f(t, 0) \equiv 0$ on I. N^c will denote the set $\|x\| \geq \rho > 0$ in R^n and I, as before, some interval $[T, \infty)$, $T \geq 0$.

DEFINITION 3. Every solution is said to be

(3.1) bounded: if, given any $t_0 \in I$ and any $r_0 > 0$, there exists an $r(t_0, r_0) > 0$ such that $\|x_0\| < r_0$ implies $\|F(t, t_0, x_0)\| < r$ for $t \geq t_0$.

(3.2) uniformly bounded: if, given any $r_0 > 0$, there exists an $r(r_0) > 0$ such that $t_0 \in I$, $\|x_0\| < r_0$ imply $\|F(t, t_0, x_0)\| < r$ for $t \geq t_0$.

(3.3) ultimately bounded: if, given any r_0, r_1, $r_0 > r_1 > 0$, there exist an $r(r_1) > 0$ and a $\tau(r_0, r_1) > 0$ such that $t_0 \in I$, $\|x_0\| < r_0$ imply $\|F(t, t, x)\| < r$ for $t \geq t_0 + \tau$.

DEFINITION 4. The solution $x = 0$ is said to be

(4.1) asymptotically stable in the large [3]: if it is stable and if $(t_0, x_0) \in I \times R^n$ implies $F(t, t_0, x_0) \longrightarrow 0$ as $t \longrightarrow \infty$.

(4.2) uniform-asymptotically stable in the large [4]: if every solution is uniformly bounded and if, given any positive r_0, r_1, there exists a $\tau(r_0, r_1) > 0$ such that $t_0 \in I$, $\|x_0\| < r_0$ imply $\|F(t, t_0, x_0)\| < r_1$ for $t \geq t_0 + \tau$.

It can be shown similarly as in the case of asymptotic stability that if $f(t, x)$ is independent of t or periodic in t on $I \times R^n$, then asymptotic stability in the large of $x = 0$ implies uniform-asymptotic stability in the large of $x = 0$, (cf., e.g., [25]).

2. We begin with theorems on boundedness which are related to some results of Yoshizawa [39], [40], [41].

THEOREM 21. If there exists, on $I \times N^c$, a real scalar function $W(t, x)$, defined, locally lipschitzian and positive definite, such that $W(t, x) \longrightarrow \infty$ with x uniformly on I and $W'(t, x) \leq 0$ on $I \times N^c$, every solution is bounded.

PROOF. Given any $t_o \in I$ and any $r_o > \rho$, let $\omega(t_o, r_o) = \sup\{W(t_o, x) \mid \rho \leq \|x\| < r_o\}$ and let $r(t_o, r_o) > \rho$ be such that $W(t, x) > \omega$ for $t \geq T$, $\|x\| \geq r$. Then $\|x_o\| < r_o$ implies $\|F(t, t_o, x_o)\| < r$ for $t \geq t_o$; for, otherwise, $\|F(t, t_o, x_o)\| = r$ at some time $t = t' > t_o$ so that $\omega < W(t', F(t', t_o, x_o)) \leq W(t_o, x_o) \leq \omega$, which is absurd.

THEOREM 22. If there exists, on $I \times N^c$, a real scalar function $W(t, x)$, defined, locally lipschitzian and positive definite, such that, for any open sphere $S \supset N$, $W(t, x)$ is bounded on $I \times N^c \cap S^c$, $W(t, x) \longrightarrow \infty$ with x uniformly on I, and $W'(t, x) \leq 0$ on $I \times N^c$, then every solution is uniformly bounded.

The proof is very similar to that of Theorem 21 and therefore is omitted.

THEOREM 23. If there exists, on $I \times N^c$, a real scalar function $W(t, x)$, defined, locally lipschitzian and positive definite, such that for any open sphere $S \supset N$, $W(t, x)$ is bounded on $I \times N^c \cap S^c$, $W(t, x) \longrightarrow \infty$ with x uniformly on I, and $W'(t, x)$ is negative definite on $I \times N^c$, then every solution is ultimately bounded.

PROOF. Given any $r_o, r_1, r_o > r_1 > \rho$, let $\omega(r_o) = \sup\{W(t, x) \mid t \in I, \rho \leq \|x\| < r_o\}$, let $\nu(r_1) > 0$ be such that $W'(t, x) \leq -\nu$ for $t \in I$, $\|x\| \geq r_1$, and put $\tau(r_o, r_1) = \omega/\nu$.

Given any $t_o \in I$ and any x_o in $\|x\| < r_o$, either $\|x_o\| < r_1$ or $r_1 \leq \|x_o\| < r_o$. In the first case, there exists an $r(r_1) \geq \rho$ such that $\|F(t, t_o, x_o)\| < r$ for $t \geq t_o$. In the second case, there exists a $t' \in [t_o, t_o + \tau]$ such that $\|F(t', t_o, x_o)\| < r_1$; for, otherwise, $\|F(t, t_o, x_o)\| \geq r_1$ throughout $[t_o, t_o + \tau]$ and hence $W(t_o + \tau, F(t_o + \tau, t_o, x_o)) \leq W(t_o, x_o) - \nu\tau \leq 0$, which is absurd. Thus, in either case, $\|F(t, t_o, x_o)\| < r$ for $t \geq t'$ and hence, a fortiori, for $t \geq t_o + \tau$.

In a series of papers [36], [37], [38] concerning second order non-linear differential equations of the Cartwright-Littlewood type, Reuter used gauge functions closely related to those of Theorem 23 to prove the ultimate boundedness of every solution and the existence of forced almost periodic solutions. He appears to have introduced the term ultimate boundedness.

3. The theorems that follow concern the uniform-asymptotic stability in the large of $x = 0$. Theorem 24 due to Massera [25] generalizes a similar theorem of Barbasin and Krasovskii [3] for the case when $f(t, x)$ is independent of t on $I \times R^n$.

THEOREM 24. If there exists, on $I \times R^n$, a Lyapunov function $V(t, x)$ such that $V(t, x) \longrightarrow 0$ and ∞ as $\|x\| \longrightarrow 0$ and ∞ uniformly on I and $V'(t, x)$ is negative definite on $I \times R^n$, $x = 0$ is uniform-asymptotically stable in the large.

The proof resembles that of Theorem 12 and therefore is omitted.

The next theorem, also due to Massera [25], is a generalization of an analogous result of Barbasin and Krasovskii [4]. Their proof hinges on Lemma 3 and requires $f(t, x)$ to be of class C^1 on $I \times R^n$ and to have bounded first partial derivatives with respect to t and x on $I \times R^n$. This result is itself an extension of an earlier theorem of theirs [3] for the case when $f(t, x)$ is independent of t on $I \times R^n$ and all solutions exist in the past; the proof of this latter theorem uses Barbasin's method of sections [2].

THEOREM 25. If $f(t, x)$ is locally lipschitzian on $I \times R^n$ and if $x = 0$ is uniform-asymptotically stable in the large, there exists, on $I \times R^n$, a Lyapunov function $V(t, x)$, possessing partial derivatives with respect to t and x of any order, such that $V(t, x) \longrightarrow 0$ and ∞ as $\|x\| \longrightarrow 0$ and ∞ uniformly on I and $V'(t, x)$ is negative definite on $I \times R^n$. If $f(t, x)$ is lipschitzian on $I \times R^n$, the partial derivatives of $V(t, x)$ are bounded on $I \times H$ for every bounded set $H \subset R^n$; if $f(t, x)$ is independent of t or periodic in t on $I \times R^n$, so is $V(t, x)$.

For the proof we refer to [25].

ADDED IN PROOF (September, 1957). Since this survey was prepared for publication, a number of papers have appeared dealing with the theory of Lyapunov's second method and, in particular, with the inversion of the basic theorems on asymptotic stability and instability. While it has been impossible to incorporate these new results, it is nevertheless felt that a few remarks about some of them are in order here.

Krasovskii [46], [48] examines several types of asymptotic stability and obtains necessary and sufficient conditions for them in terms of Lyapunov functions satisfying various additional requirements; for example, [46], he states without proof necessary and sufficient conditions for the asymptotic stability of $x = 0$ as well as for the uniform (simple) and equiasymptotic stability of $x = 0$. In [45] he gives new proofs for the converse of Theorem 13 in the case of autonomous equations and for the converse of Cetaev's theorem on instability, which was proved by Vrkoc [52]. In [47] he introduces the concept of "non-critical and uniform" behavior of a solution in an ϵ-neighborhood of $x = 0$ in terms of which he formulates necessary and sufficient criteria for the existence of Lyapunov functions in the case of asymptotic stability and instability.

Kurzweil [49] proves a converse to Theorem 15 under considerably weaker hypotheses, similar to Theorem 17 of Massera who himself extends his original method of proof, [24], to infinite dimensional spaces in [50].

Persidskii [51] and Zubov [53] consider various types of asymptotic stability, their interrelationships [53], and criteria for them involving Lyapunov functions, [51], and functions closely related to them, [53].

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IX. ON PHASE PORTRAITS OF CRITICAL POINTS IN n-SPACE*

Pinchas Mendelson

1. INTRODUCTION

In this paper we study the geometric behavior of the solution curves of the (vector) differential system

$$\frac{dx}{dt} = Ax + Q(x), \qquad x = (x_0, x_1, \ldots, x_n)$$

near the isolated critical point $x = 0$. Here A is a real, constant, $(n+1) \times (n+1)$ matrix $(n = 1, 2, 3, \ldots)$, with one zero characteristic root. All the n remaining characteristic roots have real parts of equal sign. The analytic vector function $Q(x)$ consists of higher order (non-linear) terms, subject to a certain mild restriction.

Such differential systems occur in many important stability theories (cf. Liapounov). In the two-dimensional case, $n = 1$, a rather complete analysis of the local phase portrait in the neighborhood of the origin has been carried out by several authors. Poincaré investigated the case when the matrix A is non-singular. Bendixson, Forster and Lefschetz have completed the analysis for a general matrix A in the particular case $n = 1$.

In higher dimensions Siegel has obtained significant results, but only under the restriction that A be non-singular, a restriction which excludes many of the most important applications. This paper seems to contain the first investigation of the geometrical aspect of the critical point at the origin when A is permitted to have a zero characteristic root.

The notions of "fan-cone" and "saddle-cone" are defined and the critical points of the given system are analyzed in terms of them. It is shown that in the particular case of $E_3 (n = 2)$, some major aspects of

* This work was carried out under Office of Naval Research Contract N 60 RI-105, Mathematics Dept., Princeton University, Princeton, N. J.

these new types of singularities are natural generalizations of the classical Bendixson singularities in the plane.

2. NOTATION AND TERMINOLOGY

In the following discussion x will denote a real $(n+1)$-vector with the components $(x_0, x_1, \ldots, x_n)$.

We shall use the following notations:

$[x]_h$, $(h = 1, 2, \ldots)$, to denote an $(n+1)$-vector, the components of which, $\{[x_0, x_1, \ldots, x_n]_h^i, (i = 0, 1, \ldots, n)\}$, are real power series in $(x_0, x_1, \ldots, x_n)$, convergent in some region containing the origin, and beginning with terms of degree at least h.

$^*[x_0, x_1, \ldots, x_n]_h$, $(h = 1, 2, \ldots)$, will denote a real power series in $(x_0, x_1, \ldots, x_n)$ beginning with terms of degree at least h and having no terms in x_0 alone.

$E(x_0, x_1, \ldots, x_n)$ will denote a power series in $(x_0, x_1, \ldots, x_n)$, convergent in some region containing the origin and satisfying $E(0, 0, \ldots, 0) = 1$. We shall refer to $E(x_0, x_1, \ldots, x_n)$ as a unit.

An analytic transformation $F : E_{n+1} \longrightarrow E'_{n+1}$ of the form $x'_i = f_i(x_0, x_1, \ldots, x_n)$; $f_i(0, 0, \ldots, 0) = 0$, $(i = 0, 1, \ldots, n)$, is said to be regular at the origin if the Jacobian

$$\left\| \frac{\partial f_i}{\partial x_j} \right\|_{(0,0,\ldots,0)} \neq 0 \quad .$$

Two systems of differential equations, which can be transformed into one another by an analytic transformation regular at the origin, are said to have "the same behavior" in the neighborhood of the origin (cf. Lefschetz [1] p. 118).

The following notations and definitions will be used extensively in Section 6:

(i) The double cone $\{x \mid \Sigma_{i=1}^n x_i^2 = m^2 x_0^2\}$ will be denoted by $C(m)$.

(ii) The solid cone $\{x \mid \Sigma_{i=1}^n x_i^2 \leq m^2 x_0^2\}$ will be denoted by $S(m)$.

(iii) The sets $C(m) \cap \{x \mid 0 < x_0 < \epsilon\}$ and $C(m) \cap \{x \mid -\epsilon < x_0 < 0\}$, $\epsilon > 0$, will be denoted by $C^+(m, \epsilon)$ and $C^-(m, \epsilon)$, respectively. These sets will sometimes be

referred to as "the lateral surfaces."

(iv) The sets $S(m) \cap \{x \mid 0 < x_0 \leq \epsilon\}$ and $S(m) \cap \{x \mid -\epsilon \leq x_0 < 0\}$, $\epsilon > 0$, will be denoted by $S^+(m, \epsilon)$ and $S^-(m, \epsilon)$, respectively. These sets will sometimes be referred to as "the solid cones."

(v) The sets $S(m) \cap \{x \mid x_0 = \epsilon\}$ and $S(m) \cap \{x \mid x_0 = -\epsilon\}$, $\epsilon > 0$, will be denoted by $D^+(m, \epsilon)$ and $D^-(m, \epsilon)$, respectively. We shall sometimes refer to them as "the upper and lower discs" of the cones $S^+(m, \epsilon)$ and $S^-(m, \epsilon)$.

We shall also find it useful to employ the following definitions:

DEFINITION 1. Let Φ be an autonomous system of differential equations defined in a region $\Omega \subset E_n$ and satisfying the uniqueness of solutions there. Let Γ_x, $x \in \Omega$, be the (unique) trajectory of Φ passing through x, and let A be any point set contained in Ω. Then

$$T_A = \{\Gamma_x \mid x \in A\} \quad .$$

DEFINITION 2. Let Γ_x be as defined above, and suppose that t_x is a time such that $\Gamma_x(t_x) = x$. Then $\Gamma_x^+ = \{\Gamma_x(t) \mid t > t_x\}$ and $\Gamma_x^- = \{\Gamma_x(t) \mid t < t_x\}$.

NOTE. We shall occasionally denote a point $x \in \Omega$ by Roman capitals. The symbols Γ_x, t_x, Γ_x^+ and Γ_x^- will then be written as Γ_P, t_P, Γ_P^+ and Γ_P^-, respectively.

DEFINITION 3. Let ω be an open set contained in Ω and let $B(\omega)$ be the boundary of ω. Then, we say that a point $P \in \Omega \cap B(\omega)$ <u>is a point of egress from</u> ω (with respect to the given system Φ and the set Ω), if there exists a positive number ϵ such that $\Gamma_P(t) \in \omega$ for $t_P - \epsilon < t < t_P$. If, moreover, there exists a positive number η such that $\Gamma_P(t) \in \Omega - \bar{\omega}$ for $t_P < t < t_P + \eta$, the point P is called <u>a point of strict egress from</u> ω. (A. Pliś, p. 415).

DEFINITION 4. Let Φ' be the system which is obtained from Φ by the transformation $t \longrightarrow -t$. Then $P \in \Omega \cap B(\omega)$ is <u>a point of strict access into</u> ω (with respect to the given system Φ and the set Ω) if P is a point of strict egress from ω with respect to the system Φ' and the set Ω.

DEFINITION 5. Let Φ be as defined above, and suppose that Φ has an isolated singularity at $0 \in \Omega$. The cone $S^+(m, \epsilon)$ is said to be

<u>a stable [unstable] fan-cone of</u> Φ <u>if</u>:

(i) $P \in C^+(m, \epsilon) \cup D^+(m, \epsilon) \Longrightarrow \Gamma_P^+ \subset S^+(m, \epsilon) [\Gamma_P^- \subset S^+(m, \epsilon)]$. That is, the vector field of Φ on the lateral surface and upper disc of the solid cone never points out of [into] the cone.

(ii) 0 is asymptotically stable with respect to

$$T_{S^+(m,\epsilon)} \quad \text{as} \quad t \longrightarrow +\infty \; [t \longrightarrow -\infty] \quad .$$

DEFINITION 6. If in Definition 5 every $\Gamma \in T_{S^+(m,\epsilon)}$ has a definite tangent at the origin, and if furthermore these tangents are the same for all $\Gamma \in T_{S^+(m,\epsilon)}$, then $S^+(m, \epsilon)$ will be called a <u>simple fan-cone</u>.

DEFINITION 7. $S^+(m, \epsilon)$ is said to be a <u>saddle-cone</u> of Φ (<u>of the first kind</u>) if:

(i) If $x \in C^+(m, \epsilon)$, then x is a point of strict access into the interior of $S^+(m, \epsilon)$.

(ii) $x \in C^+(m, \epsilon) \Longrightarrow \Gamma_x^+ \not\subset S^+(m, \epsilon)$.

(iii) If x is a point of the interior of $D^+(m, \epsilon)$, then x is a point of strict egress from the interior of $S^+(m, \epsilon)$. That is, all trajectories passing through the upper disc leave the solid cone (upon continuation in the positive direction).

(iv) Let A denote the set of all points $x \in D^+(m, \epsilon)$ such that $\Gamma_x^- \subset S^+(m, \epsilon)$. Then 0 is negatively asymptotically stable with respect to T_A.

DEFINITION 8. $S^+(m, \epsilon)$ is said to be a <u>saddle-cone of</u> Φ (<u>of the second kind</u>) if the transformation $t \longrightarrow -t$ reduces it to a saddle-cone of the first kind.

DEFINITION 9. $S^+(m, \epsilon)$ is said to be <u>a simple saddle-cone of</u> Φ (of the first or second kind) if the set A of Definition 7, condition (iv), consists of exactly one point.

NOTE. The above definitions were given in terms of $S^+(m, \epsilon)$ and $C^+(m, \epsilon)$, but apply, in the obvious way, also to $S^-(m, \epsilon)$ and $C^-(m, \epsilon)$.

3. A CANONICAL FORM

3.1. We begin with the system

$$\frac{dx}{dt} = Ax + [x]_2 \tag{1}$$

where

(i) $A = (a_{ij})$ is a real, constant, $(n+1) \times (n+1)$, singular matrix of rank n.

(ii) The non-zero characteristic roots of A all have negative (positive) real parts.

(iii) $\Sigma_{i=0}^{n} \dot{x}_i^2 > 0$ in $S - 0$, where S is a suitable spherical neighborhood of the origin.

A real, non-singular, constant matrix B can be found such that the transformation $x = By$ will transform the given system (1) into a system in y of the form:

$$\dot{y} = Py + [y]_2 \tag{1)'}$$

where P is in real Jordan canonical form. Furthermore, system (1)' still satisfies conditions (i), (ii), (iii). We may assume, therefore, that in the given system (1), A is already in canonical form, namely

$$A = \begin{pmatrix} 0 & 0 & 0 \cdots & 0 \\ 0 & & & \\ \cdot & & & \\ \vdots & & A_1 & \\ \cdot & & & \\ 0 & & & \end{pmatrix}$$

where A_1 is an $(n \times n)$ real, non-singular matrix in Jordan canonical form.

3.2 We observe that the components $[x_0, x_1, \ldots, x_n]_2^i$, $(i = 0, 1, \ldots, n)$, of $[x]_2$ are uniformly and absolutely convergent in some spherical neighborhood of the origin. Hence in that neighborhood these series may be rearranged. We can write, therefore:

$$[x_0, x_1, \ldots, x_n]_2^i = P_2^i(x_0) + {}^*[x_0, \ldots, x_n]_2^i, \qquad (i = 0, 1, \ldots, n) \, , \tag{2}$$

where $P_2^i(x_0)$ are power series in x_0 (with constant coefficients) starting with terms of degree at least 2.

3.3 Let $\dot{x}_i = \Sigma_{j=0}^{n} a_{ij} x_j + [x_o, \ldots, x_n]_2^i = f_i(x_o, \ldots, x_n)$, $(i = 1, \ldots, n)$, and consider the system of equations:

$$f_i(x_o, \ldots, x_n) = 0, \quad (i = 1, \ldots, n) \; . \tag{3}$$

We have

$$f_i(0, 0, \ldots, 0) = 0, \qquad (i = 1, \ldots, n) \; .$$
$$\left\| \frac{\partial f_i}{\partial x_j} \right\|_{(0,\ldots,0)} = \|A_1\| \neq 0 \; . \tag{4}$$

Hence, by the implicit function theorem, it follows that system (3) has a solution in the neighborhood of the origin, of the form:

$$x_1 = u_1(x_o), \quad x_2 = u_2(x_o), \; \ldots, \quad x_n = u_n(x_o) \tag{5}$$

where $u_i(x_o)$, $(i = 1, \ldots, n)$, are analytic functions of x_o satisfying:

$$u_i(0) = 0, \quad (i = 1, \ldots, n) \; .$$

We now return to the given system (1) and apply to it the transformation U defined by (cf. Liapounov, pp. 302-3):

$$x_o = x_o, \quad x_i = u_i + z_i, \quad (i = 1, \ldots, n) \; . \tag{6}$$

The transformation U reduces system (1) to the form:

$$\frac{dx_o}{dt} = Z_k^o(x_o) + {}^*[x_o, z_1, \ldots, z_n]_2^o \; ,$$

$$\begin{pmatrix} \frac{dz_1}{dt} \\ \cdot \\ \cdot \\ \cdot \\ \frac{dz_n}{dt} \end{pmatrix} = A_1 \begin{pmatrix} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_n \end{pmatrix} + \begin{pmatrix} Z_k^1(x_o) + {}^*[x_o, z_1, \ldots, z_n]_2^1 \\ \cdot \\ \cdot \\ \cdot \\ Z_k^n(x_o) + {}^*[x_o, z_1, \ldots, z_n]_2^n \end{pmatrix} \tag{7}$$

where $Z_k^i(x_o)$, $(i = 0, 1, \ldots, n)$, are power series in x_o beginning with terms of degree at least k, and $Z_k^o(x_o) = P_2^o(x_o) + {}^*[x_o, u_1, \ldots, u_n]_2^o$, where the right side is as defined in 3.2. Furthermore

$$(8) \qquad Z_k^i(x_o) = -\frac{du_i}{dx_o} Z_k^o(x_o), \qquad (i = 1, \ldots, n) .$$

3.4. Suppose first that $Z_k^o(x_o) \equiv 0$. Then, by virtue of (7) and (8) - if we write $(x_1, \ldots, x_n)$ for $(z_1, \ldots, z_n)$ - U transforms the given system (1) into:

$$\dot{x}_o = {}^*[x_o, \ldots, x_n]_2^o$$

$$(9) \qquad \begin{pmatrix} \dot{x}_1 \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_n \end{pmatrix} = A_1 \begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix} + \begin{pmatrix} {}^*[x_o, x_1, \ldots, x_n]_2^1 \\ \cdot \\ \cdot \\ \cdot \\ {}^*[x_o, x_1, \ldots, x_n]_2^n \end{pmatrix}$$

The transformation U is analytic and regular at the origin. Hence, in the neighborhood of the origin the behavior of the trajectories of system (9) is the same as that of system (1). We note, however, that $x_1 = x_2 = \ldots = x_n = 0$, $x_o = c$ is a solution of (9) for any real constant c. Thus in (9), and hence in (1), the origin is not an isolated singularity, contradicting condition (iii).

It follows therefore that $Z_k^o \not\equiv 0$.

3.5. The above considerations have transformed the given system (1) into one of the form:

$$\dot{x}_o = X_k^o(x_o) + {}^*[x_o, x_1, \ldots, x_n]_2^o$$

$$(10) \qquad \begin{pmatrix} \dot{x}_1 \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_n \end{pmatrix} = A \begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix} + \begin{pmatrix} X_k^1(x_o) + {}^*[x_o, x_1, \ldots, x_n]_2^1 \\ \cdot \\ \cdot \\ \cdot \\ X_k^n(x_o) + {}^*[x_o, x_1, \ldots, x_n]_2^n \end{pmatrix}$$

where,

(i) A is an $(n \times n)$, real, non-singular matrix in real Jordan canonical form.

(ii) $X_k^o(x_o) = \Sigma_{j=k}^{\infty} c_j x_o^j$, $c_k \neq 0$, $k \geq 2$.

(iii) $X_k^i(x_o)$, $(i = 1, \ldots, n)$, are real power series in x_o beginning with terms of degree at least k.

REMARK. The matrix A appearing in (10) is the matrix A_1 of 3.1.

3.6. Writing $X_k^o(x_o) = c_k x_o^k E(x_o)$, where $E(x_o)$ is a unit, we apply the transformation $t \longrightarrow t_1$ defined by:

$$dt_1 = c_k E(x_o) dt$$

thereby transforming (10) into

$$\frac{dx_o}{dt_1} = x_o^k + {}^*[x_o, x_1, \ldots, x_n]_2^o \, c_k^{-1} E_1(x_o)$$

$$(11) \quad \begin{pmatrix} \frac{dx_1}{dt_1} \\ \vdots \\ \frac{dx_n}{dt_1} \end{pmatrix} = A \begin{pmatrix} x_1 c_k^{-1} E_1(x_o) \\ \vdots \\ x_n c_k^{-1} E_1(x_o) \end{pmatrix} + c_k^{-1} E_1(x_o) \begin{pmatrix} X_k^1(x_o) + {}^*[x_o, \ldots, x_n]_2^1 \\ \vdots \\ X_k^n(x_o) + {}^*[x_o, \ldots, x_n]_2^n \end{pmatrix}$$

where $E_1(x_o) = E(x_o)^{-1}$ is also a unit.

NOTE. Since $E(x_o)$ is a unit, there exists a small enough neighborhood S of the origin such that $E(x_o) > \frac{1}{2}$ throughout S. Let now F_1 be the vector field defined by system (10) and let F_2 be the vector field defined by system (11). Then F_1 and F_2 differ throughout S only by a continuous scalar factor (namely, $c_k E(x_o)$) which is positive in case $c_k > 0$ and negative in case $c_k < 0$. The fields define two systems of trajectories throughout S which are identical, except for the fact that in the case $c_k < 0$ the sense of the trajectories is reversed.

Rearranging terms, system (11) becomes:

$$\dot{x}_o = x_o^k + {}^*[x_o, \ldots, x_n]_2^o$$

$$(12) \quad \begin{pmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_n \end{pmatrix} = A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} P_k^1(x_o) + {}^*[x_o, \ldots, x_n]_2^1 \\ \vdots \\ P_k^n(x_o) + {}^*[x_o, \ldots, x_n]_2^n \end{pmatrix}$$

where A is as in (10) and $P_k^i(x_o)$, $(i = 1, \ldots, n)$, are power series in x_o beginning with terms of degree at least k.

3.7. It follows from the above discussion that system (12) is a canonical form for real analytic systems of differential equations having an isolated singularity with one zero characteristic root at the origin. We shall assume henceforth that the given system has already been reduced to that form.

3.8. The study of the most general systems of the type (1) is rather complicated. We shall therefore restrict our attention to systems (1) whose canonical form (12) satisfies:

(iv) $\dot{x}_0 = x_0^k + {}^*[x_0, \ldots, x_n]_{k+1}^0$
i.e., x_0^k is the term of least degree in the expression for $\dot{x}_0$.

Geometrically this added condition implies that the surfaces $\dot{x}_0 = x_0^k + {}^*[x_0, \ldots, x_n]_{k+1}^0 = 0$, when they exist, are tangent to the $x_0 = 0$ plane at the origin.

4. THE TRANSFORMATION T

4.1. Let x' denote points in a second $(n + 1)$ dimensional space E'_{n+1}. Let $T : E_{n+1} \longrightarrow E'_{n+1}$ be defined by:

$$x_0 = x_0', \quad x_i = x_i' x_0', \quad (i = 1, 2, \ldots, n) \quad . \tag{13}$$

Then T can be thought to map the origin 0 of E_{n+1} onto the $x_0' = 0$ plane, and all other points of the $x_0 = 0$ plane into the point at infinity of the plane $x_0' = 0$. Outside $x_0 = 0$ the map is one-to-one and analytic at every point.

4.2. For $x_0 \neq 0$ we have:

$$x_0' = x_0, \quad x_i' = x_i x_0^{-1}, \quad (i = 1, \ldots, n)$$

and hence

$$\dot{x}_0' = \dot{x}_0, \quad \dot{x}_i' = \dot{x}_i x_0^{-1} - x_i x_0^{-2} \dot{x}_0, \quad (i = 1, 2, \ldots, n) \quad .$$

The transform of (12) by T yields:

$$\begin{aligned} \dot{x}_0' &= (x_0')^k + {}^*[x_0', x_1' x_0', \ldots, x_n' x_0']_{k+1}^0, \qquad k \geq 2 \\ \dot{x}_i' &= \sum_{j=1}^{n} a_{ij} x_j' + c_i x_0' + x_0'[x_0', x_1', \ldots, x_n']_1^1, \quad (i = 1, \ldots, n). \end{aligned} \tag{14}$$

The term $c_i x_0'$ can only appear in the case $k = 2$, but in that case we apply a suitable linear transformation which reduces system (14) further to:

$$
\begin{aligned}
\dot{x}_0' &= (x_0')^k + {}^*[x_0', x_1'x_0', \ldots, x_n'x_0']^0_{k+1}, \qquad k \geq 2 \\
(15) \qquad \begin{pmatrix} \dot{x}_1' \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_n' \end{pmatrix} &= A \begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix} + \begin{pmatrix} x_0'[x_0', \ldots, x_n']^1_1 \\ \cdot \\ \cdot \\ \cdot \\ x_0'[x_0', \ldots, x_n']^n_1 \end{pmatrix}
\end{aligned}
$$

4.3. Since

$$ {}^*[x_0', x_1'x_0', \ldots, x_n'x_0']^0_{k+1} = (x_0')^{k+1*}[x_0', x_1', \ldots, x_n']^0_1 $$

it follows that

$$ (16) \qquad \dot{x}_0' = (x_0')^k + (x_0')^{k+1*}[x_0', x_1', \ldots, x_n']^0_1 = (x_0')^k E(x_0', \ldots, x_n') \quad . $$

The transformation $t \longrightarrow t_1$ defined by $dt_1 = E(x_0', \ldots, x_n')dt$ reduces system (15) further to the form:

$$
\begin{aligned}
\dot{x}_0' &= (x_0')^k , \qquad\qquad k \geq 2 \\
(17) \qquad \begin{pmatrix} \dot{x}_1' \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_n' \end{pmatrix} &= A \begin{pmatrix} x_1' \\ \cdot \\ \cdot \\ \cdot \\ x_n' \end{pmatrix} + x_0' \begin{pmatrix} [x_0', \ldots, x_n']^1_1 \\ \cdot \\ \cdot \\ \cdot \\ [x_0', \ldots, x_n']^n_1 \end{pmatrix}
\end{aligned}
$$

4.4. Let R be a ray emanating from the origin 0 and contained in $x_0 > 0$ $[x_0 < 0]$. Then R can be represented by the set of $(n + 1)$ relations:

$$ x_0 > 0 \ [x_0 < 0], \quad x_1 = m_1x_0, \quad x_2 = m_2x_0, \ \ldots, \quad x_n = m_nx_0 \quad . $$

Let $R' = T(R)$. Then clearly R' is the half-line:

$$ x_0' > 0 \ [x_0' < 0], \quad x_1' = m_1, \quad x_2' = m_2, \ \ldots, \quad x_n' = m_n \quad . $$

i.e., the half line perpendicular to the $x_0' = 0$ plane and intersecting that plane at the point $(0, m_1, m_2, \ldots, m_n)$.

Furthermore, if $C^+(m, \epsilon)$ is the half cone defined above, then $T(C^+(m, \epsilon))$ is the half cylinder $\{x' \mid 0 < x_0' < \epsilon, \ \Sigma_{i=1}^n x_i'^2 = m^2\}$. Similarly, $T(C^-(m, \epsilon)) = \{x' \mid -\epsilon < x_0' < 0, \ \Sigma_{i=1}^n x_i'^2 = m^2\}$.

4.5. Let now $r(t)$ be a trajectory of (12) which is contained in $C^+(m, \epsilon)$ $[C^-(m, \epsilon)]$ and let $r'(t)$ be the image of $r(t)$ under T.

Then $\Gamma'(t)$ is contained in the cylinder $T(C^+(m, \epsilon))$ $[T(C^-(m, \epsilon))]$; and if $\Gamma(t)$ tends to 0 in E_{n+1} for $t \longrightarrow +\infty$ $(t \longrightarrow -\infty)$, then $\Gamma'(t)$ tends to $x_0' = 0$ in E_{n+1}' (i.e., $\Gamma'(t)$ tends to the base of the cylinder). Moreover, if $\Gamma(t)$ tends to 0 with a definite tangent $R : x_1 = m_1 x_0, \ldots, x_n = m_n x_0$, then $\Gamma'(t)$ tends to the point $(0, m_1, m_2, \ldots, m_n)$. In particular, if $\Gamma(t)$ tends to the origin tangent to the x_0-axis, then $\Gamma'(t)$ tends to the origin $0'$.

Conversely, if $\Gamma'(t)$ is contained in a half cylinder, as above, then $\Gamma(t)$ is contained in the corresponding cone. If $\Gamma'(t)$ tends to the $x_0' = 0$ plane (inside the cylinder), then $x_i = x_i' x_0' \leq m x_0' \longrightarrow 0$ as $x_0' \longrightarrow 0$, and hence $\Gamma(t)$ tends to the origin. Finally, if $\Gamma'(t) \longrightarrow (0, m_1, \ldots, m_n)$ as $t \longrightarrow \pm\infty$, then $\Gamma(t)$ has a definite tangent at the origin, this tangent being the line: $x_1 = m_1 x_0, \ldots, x_n = m_n x_0$.

We shall find it convenient to study system (17) in detail and obtain thereby, in view of the above considerations, information about the system (12).

5. PHASE PORTRAIT OF SYSTEM (17)

5.1. Replacing $(x_0', \ldots, x_n')$ by $(x_0, \ldots, x_n)$ and $[x_0', \ldots, x_n']_1^i$, $(i = 1, \ldots, n)$, by $X_i = X_i(x_0, \ldots, x_n)$ we may rewrite (17) as:

$$\dot{x}_0 = x_0^k, \qquad k \geq 2$$

$$(18) \qquad \begin{pmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_n \end{pmatrix} = A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + x_0 \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} .$$

The matrix A appearing in (18) is in real Jordan canonical form. We may assume, without loss of generality, that

$$(19) \qquad A = \begin{pmatrix} A_1 & 0 & \cdots & & & 0 \\ 0 & A_2 & \cdots & & & 0 \\ \vdots & & \ddots & & & \\ 0 & \cdots & & A_r & \cdots & 0 \\ 0 & \cdots & & 0 & B_1 \cdots & 0 \\ \vdots & & & & \ddots & \vdots \\ 0 & \cdots & & & & B_s \end{pmatrix}$$

the A_i, B_j, $(i = 1, \ldots, r)$, $(j = 1, \ldots, s)$, being real square matrices of α_i and β_j rows, respectively. Each A_i is of the form

$$(20) \qquad A_i = \begin{pmatrix} \lambda_i & 0 & \cdots & & 0 \\ \xi & \lambda_i & \cdots & & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & & \lambda_i & 0 \\ 0 & \cdots & & \xi & \lambda_i \end{pmatrix}$$

where ξ is any arbitrarily chosen real positive number, and each B_j is of the form

$$(21) \qquad B_j = \begin{pmatrix} M_j & 0 & \cdots & & 0 \\ E_2 & M_j & \cdots & & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & & M_j & 0 \\ 0 & \cdots & & E_2 & M_j \end{pmatrix}$$

where M_j is the real matrix

$$\begin{pmatrix} \mu_j & -\nu_j \\ \nu_j & \mu_j \end{pmatrix} \quad \text{and} \quad E_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} .$$

5.2. The following cases will be dealt with separately:

CASE 1. $k = 2p, \quad p \geq 1$

A. $\lambda_i < 0, \quad (i = 1, \ldots, r); \quad \mu_j < 0, \quad (j = 1, \ldots, s)$.

B. $\lambda_i > 0, \quad (i = 1, \ldots, r); \quad \mu_j > 0, \quad (j = 1, \ldots, s)$.

CASE 2. $k = 2p + 1, \quad p \geq 1$

A. $\lambda_i < 0, \quad (i = 1, \ldots, r); \quad \mu_j < 0, \quad (j = 1, \ldots, s)$.

B. $\lambda_i > 0, \quad (i = 1, \ldots, r); \quad \mu_j > 0, \quad (j = 1, \ldots, s)$.

5.3. CASE 1A.

5.3.1. We note first that in system (18), $x_0 = 0$ satisfies the first equation, and consequently the $(x_1, \ldots, x_n)$ hyperplane is an integral surface.

5.3.2. Let

$$\rho^2 = \sum_{i=1}^{n} x_i^2 .$$

Then

$$
\begin{aligned}
\rho\dot{\rho} &= \sum_{i=1}^{n} x_i \dot{x}_i \\
&= \lambda_1 \left(x_1^2 + \ldots + x_{\alpha_1}^2 \right) \\
&\quad + \lambda_2 \left(x_{\alpha_1+1}^2 + \ldots + x_{\alpha_1+\alpha_2}^2 \right) + \\
&\qquad \cdot \quad \cdot \quad \cdot \\
&\quad + \lambda_r \left(x_{\alpha_1+\ldots+\alpha_{r-1}+1}^2 + \ldots + x_{\alpha_1+\ldots+\alpha_r}^2 \right) \\
&\quad + \mu_1 \left(x_{\alpha_1+\ldots+\alpha_r+1}^2 + \ldots + x_{\alpha_1+\ldots+\alpha_r+\beta_1}^2 \right) + \\
&\qquad \cdot \quad \cdot \quad \cdot \\
&\quad + \mu_s \left(x_{\alpha_1+\ldots+\alpha_r+\beta_1+\ldots+\beta_{s-1}+1}^2 + \ldots + x_n^2 \right) \\
&\quad + \xi \sum x_i x_j \\
&\quad + x_0 \sum_{i=1}^{n} x_i X_i \quad .
\end{aligned} \tag{22}
$$

We note that there are at most $(n - 1)$ pairs (i, j) such that the corresponding products $x_i x_j$ appear in the expression $\xi \Sigma x_i x_j$. Since for all (i, j) $|x_i x_j| \leq \rho^2$, and since $\xi > 0$, we can write immediately:

$$
|\xi \sum x_i x_j| \leq n \xi \rho^2 \tag{23}
$$

5.3.3. Let

$$
2\lambda = \underset{\substack{i=1,\ldots,r \\ j=1,\ldots,s}}{\mathrm{Min.}} \left\{ |\lambda_i|, \ |\mu_j| \right\} \quad .
$$

Then $\lambda > 0$, and it follows from (22), (23) that

(24) $$\rho\dot{\rho} \leq -2\lambda\rho^2 + n\xi\rho^2 + x_o \sum_{i=1}^{n} x_i X_i \quad .$$

Since ξ can be chosen at will, we select $0 < \xi < \frac{\lambda}{n}$. Then $0 < \xi n\rho^2 < \lambda\rho^2$ and hence,

(25) $$\rho\dot{\rho} < -\lambda\rho^2 + x_o \sum_{i=1}^{n} x_i X_i \quad .$$

Let $x_i = \theta_i(x)\rho$, $(i = 1, \ldots, n)$. Then $|\theta_i(x)| \leq 1$ for all x, and we have:

$$\sum_{i=1}^{n} x_i X_i = \rho \sum_{i=1}^{n} \theta_i(x) X_i = \rho \sum_{i=1}^{n} R_i(x) \quad ,$$

where $R_i(x) = \theta_i(x) X_i$.

Therefore,

(26) $$\rho\dot{\rho} < -\lambda\rho^2 + \rho\,|x_o| \sum_{i=1}^{n} |R_i| \quad ,$$

or,

(27) $$\dot{\rho} < -\lambda\rho + |x_o| \sum_{i=1}^{n} |R_i| \quad .$$

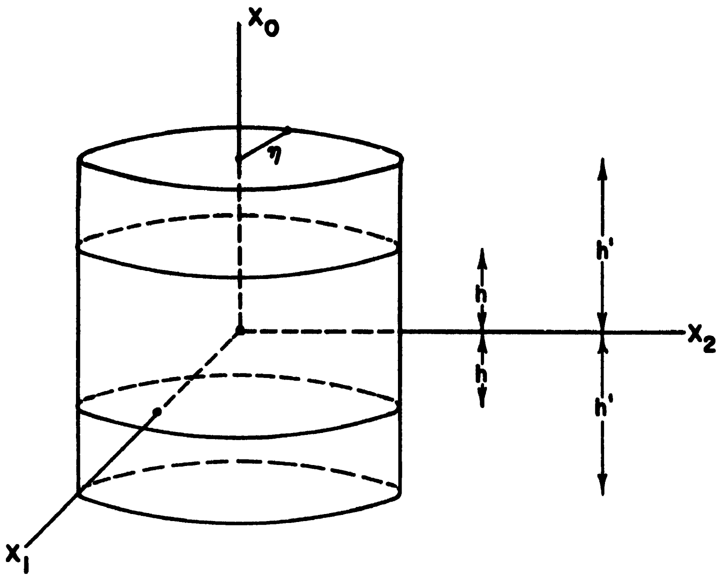
Let S be the common domain of convergence of $R_1, \ldots, R_n$ and let C' be a closed cylinder of height $2h'$ and radius η which is contained in S (Figure 1).

Furthermore, let

$$M = \underset{x \in C'}{\mathrm{Max.}} \left(\sum_{i=1}^{n} |R_i| \right) \quad .$$

Let C be another cylinder of radius η and height $2h$, where

$$h < \frac{\lambda\eta}{2M} \,, \quad h \leq h' \quad \text{(Figure 1)} \quad .$$



$$C' = \left\{ \sum_{i=1}^{n} x_i^2 \le \eta^2; \quad -h' \le x_0 \le h' \right\}$$

$$C = \left\{ \sum_{i=1}^{n} x_i^2 \le \eta^2; \quad -h \le x_0 \le h \right\}$$

FIGURE I

Obviously

$$\operatorname*{Max.}_{x\epsilon C} \left(\sum_{i=1}^{n} |R_i| \right) \le \operatorname*{Max.}_{x\epsilon C'} \left(\sum_{i=1}^{n} |R_i| \right) = M \quad .$$

Therefore, for points on the lateral surface of C, we have from (27):

$$\dot{\rho} < -\lambda\eta + hM \leq -\lambda\eta + \frac{\lambda\eta}{2M} \cdot M = -\frac{\lambda\eta}{2} < 0 \quad . \tag{28}$$

Hence, on the lateral surface of C, the vector field defined by (18) points into the cylinder C. Note that in the case being discussed $k = 2p$ and therefore $x_0^k > 0$ for $x_0 \neq 0$. Hence, on the bottom disc the vector field points into C, whereas on the top disc the vector field points out of C.

5.3.4. Consider the set

$$C_1 = C \cap \left\{(x_0, \ldots, x_n) \mid x_0 < 0\right\} \tag{29}$$

and let

$$C_1' = \left\{x \mid -h < x_0 < 0, \sum_{i=1}^{n} x_i^2 = \eta^2\right\} \cup \left\{x \mid x_0 = -h, \sum_{i=1}^{n} x_i^2 \leq \eta^2\right\} . \tag{30}$$

(C_1' is the open lateral surface of the lower half of C together with the closed lower disc of C.) Recalling that the $(x_1, \ldots, x_n)$ hyperplane is a solution surface, the above results imply that if Γ_P is any trajectory passing through $P \in C_1'$ at time t_P, then $\Gamma_P \subset C_1$ for all $t > t_P$. Since $\dot{x}_0 = x_0^{2p} > 0$ throughout C_1, Γ_P must rise monotonically towards the $(x_1, \ldots, x_n)$ plane and cannot have any point $Q \in C_1$ in its positive limit set $\Lambda^+(\Gamma_P)$. Thus, the positive limit set of Γ_P is contained in the disc

$$D: x_0 = 0, \quad \sum_{i=1}^{n} x_i^2 \leq \eta^2 \quad .$$

However, if $\Lambda^+(\Gamma_P) \ni P_1 \in D$, where $P_1 = (0, \zeta_1, \ldots, \zeta_n)$ with

$$\sum_{i=1}^{n} \zeta_i^2 = \zeta > 0 \quad ,$$

an argument identical with the one used previously will show that there exists an $h(\zeta)$ such that for $|x_0| < h(\zeta)$, (i.e., for t large enough) $\dot{\rho} < 0$ in the neighborhood of P_1. Thus $P_1 \notin \Lambda^+(\Gamma_P)$. Hence $\Lambda^+(\Gamma_P)$ consists of the origin only.

Thus all trajectories entering C_1 tend to the origin in the direction of increasing time. It will be shown later that they do so

tangent to the x_o-axis. The phase-portrait is that of a "fan" tangent to the x_o-axis as indicated in Figure 3.

5.3.5. Consider next the set $C_2 = C \cap \{x \mid x_o > 0\}$. Let $P(\zeta_o, \zeta_1, \ldots, \zeta_n)$ be a point on its open lateral surface and let Γ_P be the trajectory passing through P at time t_P. We have seen before that Γ_P must enter C_2 at P. Since $\dot{x}_o = x_o^{2p} > 0$ is monotonic, $\dot{x}_o > \zeta_o^{2p} > 0$ for all points of Γ_P at time $t > t_P$. Hence Γ_P leaves C_2 after a finite time τ_P, and must do so through the upper disc. We denote the first point of departure by $\tilde{\tilde{P}}$.

Let f be the mapping of the lateral surface of C_2 into the upper disc defined by:

$$f : P \longrightarrow \tilde{\tilde{P}} .$$

Since Γ_P traverses its path from P to $\tilde{\tilde{P}}$ in a time τ_P which is finite, it follows that f is a homeomorphism.

Let $P_m(\zeta_m, \eta, 0, \ldots, 0)$, $(m = 0, 1, \ldots)$, be a sequence of points on the lateral surface of C_2 such that:

$$\zeta_o = \frac{h}{2}, \quad \zeta_m \longrightarrow 0 \quad \text{as} \quad m \longrightarrow \infty ,$$

and let J_m, $(m = 0, 1, \ldots,)$, be the $(n-1)$ sphere defined by

$$\sum_{i=1}^{n} x_i^2 = \eta^2, \quad x_o = \zeta_m \quad \text{(Figure 2)} .$$

Then $f(J_m) = J_m'$ is contained in the upper disc of C_2. Let A_m, $(m = 0, 1, 2, \ldots)$, denote J_m' and its interior. Then A_m is a closed bounded set lying in the upper disc and $A_m \supset A_{m+1}$ for $m = 0, 1, 2, \ldots$ Hence

$$A = \bigcap_{m=0}^{\infty} A_m \neq \emptyset .$$

Let $Q \in A$ and consider the trajectory Γ_Q passing through Q at time t_Q. We shall prove that Γ_Q stays in C_2 for all time $t < t_Q$. For suppose Γ_Q, when continued in the negative direction, leaves C_2 at some point P. Then $P(\eta_o, \eta_1, \ldots, \eta_n)$ must be on the open lateral surface of C_2, and thus $\eta_o > 0$. Hence $\eta_o > \zeta_m$ for some m implying that $Q \notin A_m$. It follows that $Q \notin A$, contradicting our original

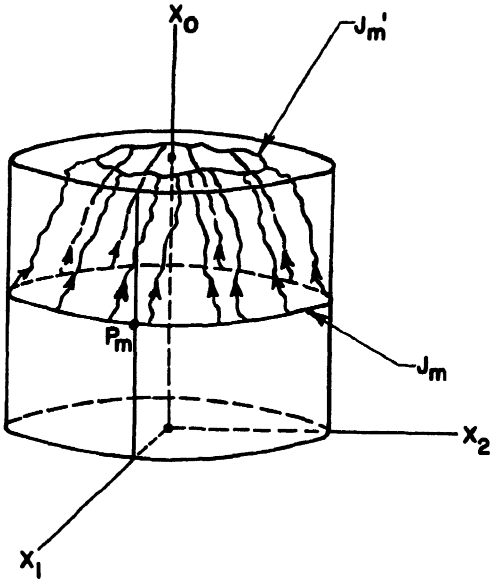


FIGURE 2

assumption. Therefore we must conclude that Γ_Q remains in C_2 for all $t \leq t_Q$.

Using an argument similar to the one used with respect to trajectories approaching the origin from below the $(x_1, \ldots, x_n)$ hyperplane, we can show that the negative limit set of Γ_Q consists solely of the singular point at the origin.

We have thus proved:

LEMMA 1. System (18), case 1A, has at least one trajectory tending to the origin from above the $(x_1, \ldots, x_n)$ plane in the negative direction.

We now proceed to prove:

LEMMA 2. System (18), case 1A, has only one trajectory tending to the origin from above the $(x_1, \ldots, x_n)$ plane in the negative direction.

PROOF. Assume there are two such trajectories and let them be

denoted by $(x_0^1, x_1^1, \ldots, x_n^1)$ and $(x_0^2, x_1^2, \ldots, x_n^2)$, respectively. Let $x_i^2 - x_i^1$, $(i = 1, 2, \ldots, n)$, be denoted by u_i and define,

$$\rho^2 = \sum_{i=1}^{n} u_i^2 \ . \tag{31}$$

Then replacing $x_0 X_i(x_0, \ldots, x_n)$ in (18) by $X_i'(x_0, \ldots, x_n)$, $(i = 1, \ldots, n)$, and following a calculation similar to one carried out above (5.3.3), we get:

$$\rho\dot{\rho} \leq -\lambda\rho^2 + \sum_{i=1}^{n} u_i\left[X_i'\left(x_0, x_1^2, \ldots, x_n^2\right) - X_i'\left(x_0, x_1^1, \ldots, x_n^1\right)\right]$$

$$= -\lambda\rho^2 + \sum_{i=1}^{n} u_i\left[\sum_{j=1}^{n} \frac{\partial X_i'}{\partial x_j} u_j + \epsilon_i\right]$$

$$\leq -\lambda\rho^2 + \rho^2 \sum_{i,j=1}^{n} \left|\frac{\partial X_i'}{\partial x_j}(x_0, x_1^1, \ldots, x_n^1)\right| + \tag{32}$$

$$+ \sum_{i=1}^{n} \epsilon_i \, |u_i| \quad .$$

Since X_i', $(i = 1, \ldots, n)$, are power series beginning with terms of degree at least 2, it follows immediately that by choosing a small enough neighborhood of the origin we can make

$$\rho\dot{\rho} \leq -\frac{\lambda}{2}\rho^2 \tag{33}$$

or,

$$\dot{\rho} \leq -\frac{\lambda}{2}\rho \quad . \tag{34}$$

Since $\dot{x}_0 = x_0^{2p} > 0$ and $\frac{d\rho}{dt} < 0$, we get:

$$\frac{d\rho}{dx_0} = \frac{d\rho}{dt} \cdot \frac{dt}{dx_0} < 0 \quad ,$$

which is absurd. The uniqueness of the trajectory of Lemma 1 is therefore established.

The phase portrait of a system such as (18) in E_3 with

$$A = \begin{pmatrix} -\lambda & 0 \\ 0 & -\mu \end{pmatrix}$$

and $k = 2p$ is given in Figure 3.

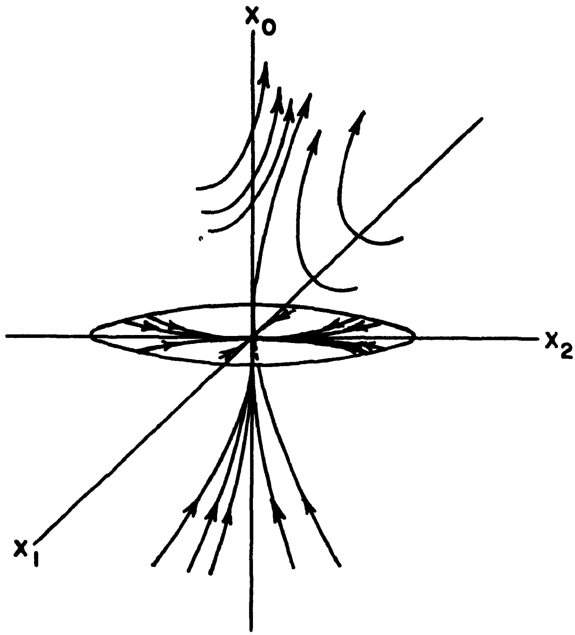


FIGURE 3

The singularity shown in Figure 3 is a generalization to three dimensions of a Bendixson singularity (of the second type) with one nodal and two hyperbolic sectors (Figure 4). In fact, if we rotate the singularity of Figure 4 around the x_0-axis, we get a special case in E_3 of the singularity treated above.

5.4. Case 1B

In this case $k = 2p$, $p \geq 1$ and all roots of A have positive real parts. The double transformation $t \longrightarrow -t$, $x_0 \longrightarrow -x_0$ reduces this system to one of the form 1A.

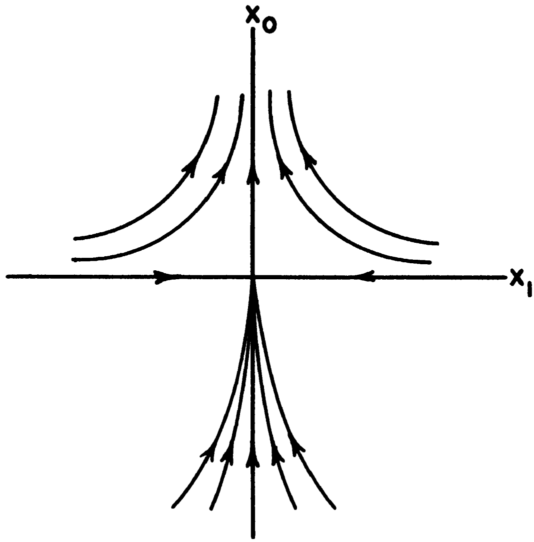


FIGURE 4

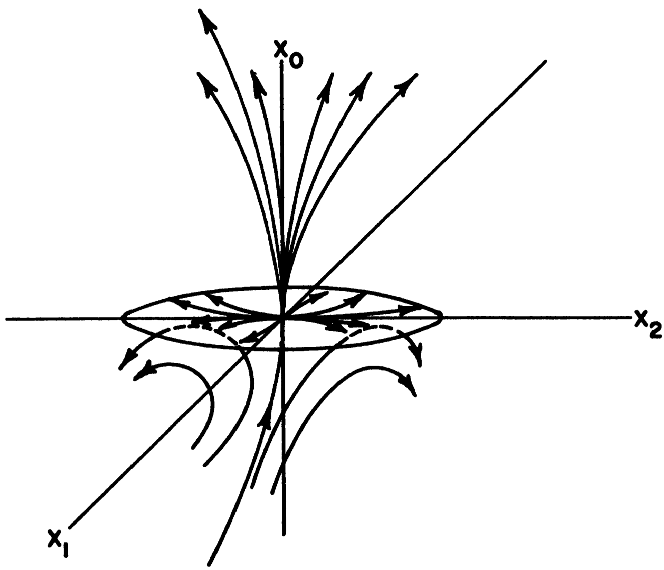


FIGURE 5

A system (18) in E_3 with $k = 2p$ and

$$A = \begin{pmatrix} +\lambda & 0 \\ 0 & +\mu \end{pmatrix}$$

will have the phase portrait indicated in Figure 5.

5.5. Case 2A

In this case $k = 2p + 1$ and all the roots of A have negative real parts. We note that the transformation $x_0 \longrightarrow -x_0$ transforms the system to a new system of the same general type. Hence, system (18) in case 2A exhibits the same phase portrait above and below the $(x_1, \ldots, x_n)$ plane. Furthermore, our treatment of systems of the types 1A above and on the $(x_1, \ldots, x_n)$ plane applies completely to systems of the type 2A in the same region. The even power of x_0 appearing in the expression for $\dot{x}_0$ in 1A was only used to determine the sense of the x_0 derivative, and this sense is the same for $x_0 \geq 0$ for both even and odd powers. The phase portrait of a system of this type in E_3 with

$$A = \begin{pmatrix} -\lambda & 0 \\ 0 & -\mu \end{pmatrix}$$

is given in Figure 6.

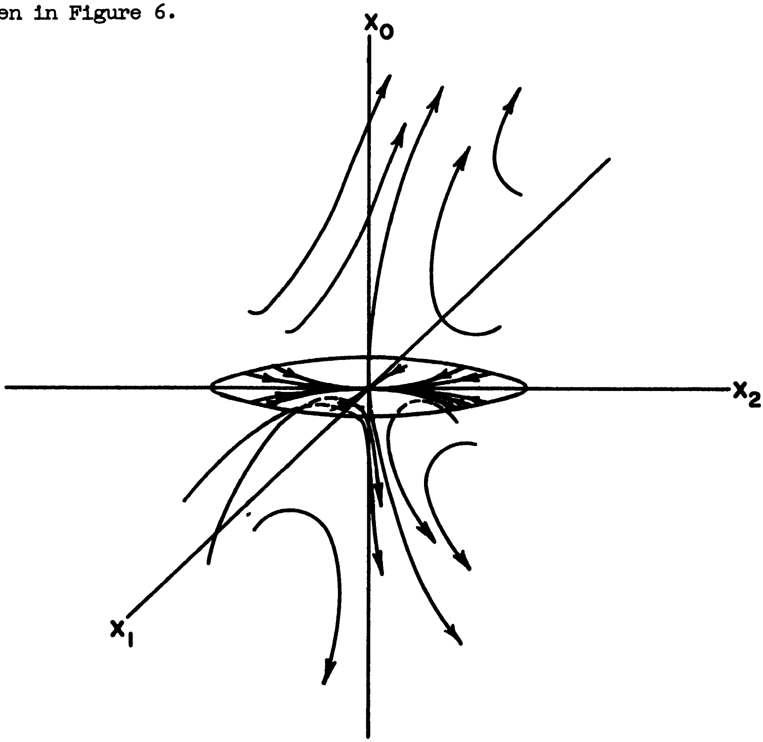


FIGURE 6

This singularity (Figure 6) is a generalization to three dimensions of a Bendixson singularity (of the second type) with four hyperbolic sectors. The rotation of Figure 7 around the x_0-axis produces a special case of Figure 6.

5.6. Case 2B

Again, this system exhibits the same phase portrait above and below the $(x_1, \ldots, x_n)$ plane. Its behavior above and on the $(x_1, \ldots, x_n)$ plane is the same as that of 1B in the same region. The phase portrait in E_3 with

$$A = \begin{pmatrix} +\lambda & 0 \\ 0 & +\mu \end{pmatrix}$$

is given in Figure 8.

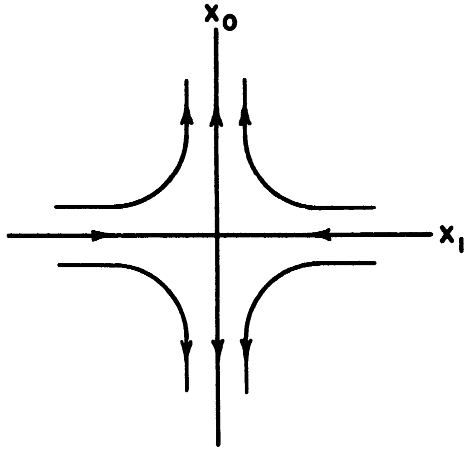


FIGURE 7

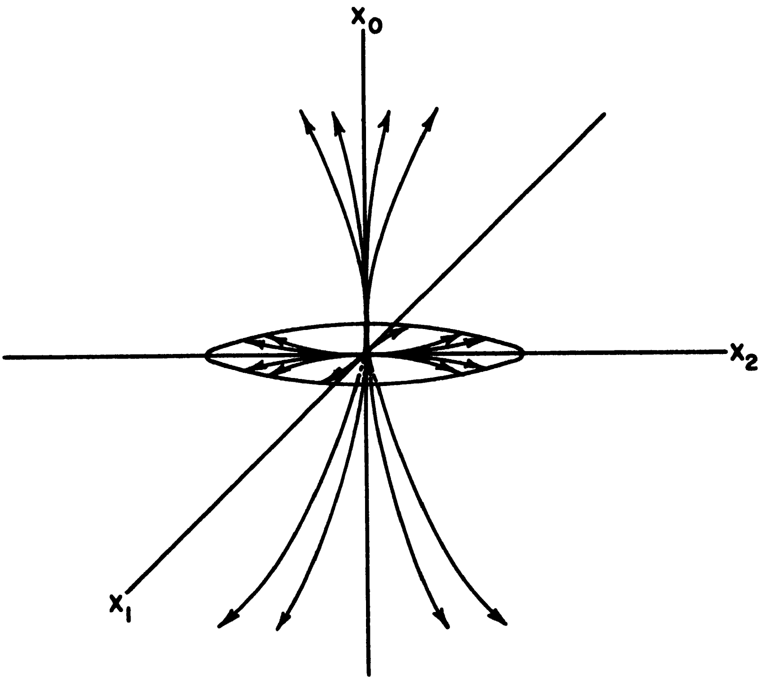


FIGURE 8

6. CONCLUSION

6.1. Let $P_k^i(x_0) + {}^*[x_0, \ldots, x_n]_2^i$ in system (12) be denoted by $X_i(x_0, \ldots, x_n)$, $(i = 1, \ldots, n)$, and let $D \subset E_{n+1}$ be contained in the domain of convergence of $X_i(x_0, \ldots, x_n)$, $(i = 1, \ldots, n)$. Choose $\xi > 0$ small enough so that

$$S = \left\{ x \mid 0 < x_0 < \xi, \quad \sum_{i=1}^{n} x_i^2 < \xi^2 \right\}$$

is contained in D. The transformation T defined in 4.1 maps S into the set $T(S) \subset E'_{n+1}$, where

$$T(S) = \left\{ (x_0', \ldots, x_n') \;\middle|\; 0 < x_0' < \xi; \quad 0 < x_0' \rho' < \xi \right\} ,$$

where

$$\rho' = \left[x_1'^2 + x_2'^2 + \ldots + x_n'^2 \right]^{1/2}$$

Since system (17) is the transform of system (12) by T, it follows that T(S) is contained in the common domain of convergence of the $[x_0', \ldots, x_n']_1^i$, $(i = 1, \ldots, n)$, appearing in (17).

Given any $\eta > 0$ there exists an $\epsilon(\eta) > 0$ small enough so that the cylinder

$$\left\{ x' \;\middle|\; 0 < x_0' \leq \epsilon(\eta), \quad \sum_{i=1}^{n} x_i'^2 = \eta^2 \right\}$$

is contained in T(S). Thus, for any η, there exists a cylinder low enough that the discussion of Section 5 applies to it. The situation is the same for the lower half space, $x_0 < 0$.

The above remarks together with the discussion contained in Sections 4 and 5 now complete the proof of the following theorems:

THEOREM 1. Let there be given a system (1) in canonical form (12) and satisfying conditions (i) - (iv), of Sections 3.1 and 3.8. Then for any $m > 0$ there exists an $\epsilon(m) > 0$ such that $S^+(m, \epsilon(m))$ and $S^-(m, \epsilon(m))$ have the following properties:

	$k = 2p, \quad p \geq 1$		$k = 2p + 1, \quad p \geq 1$	
Real Parts of Roots of A	Negative	Positive	Negative	Positive
$S^+(m, \epsilon(m))$	simple saddle-cone (of the first kind	unstable, simple fan-cone	simple saddle-cone (of the first kind)	unstable, simple fan-cone
$S^-(m, \epsilon(m))$	stable, simple fan-cone	simple saddle-cone (of the second kind)	simple saddle-cone (of the first kind)	unstable, simple fan-cone

THEOREM 2. In Theorem 1, all trajectories which tend to the origin in the positive or negative direction, either from the fan-cones or from the saddle-cones, are tangent to the x_0-axis at the origin.

The local phase portraits of these systems will exhibit the characteristics shown in Figures 9, 10, 11 and 12.

REMARK. Since system (17) is a special case of system (12) it follows that Theorems 1 and 2 apply to it as well. The phase portraits of these particular systems are, therefore, completely described.

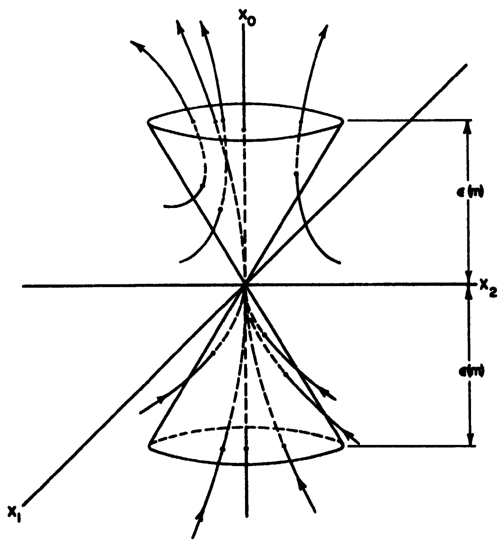


FIGURE 9

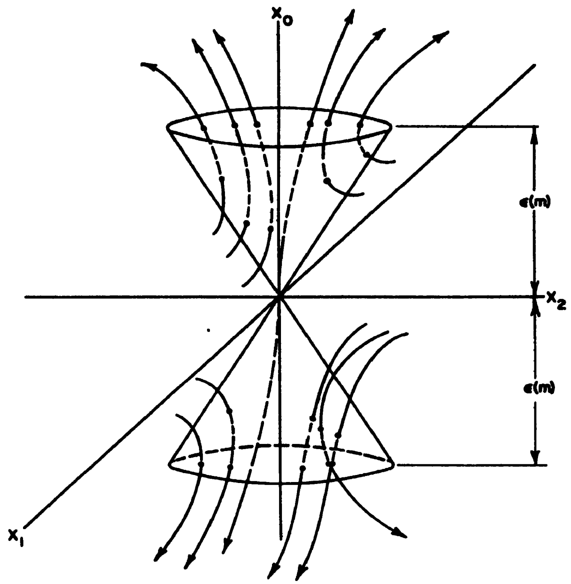


FIGURE 10

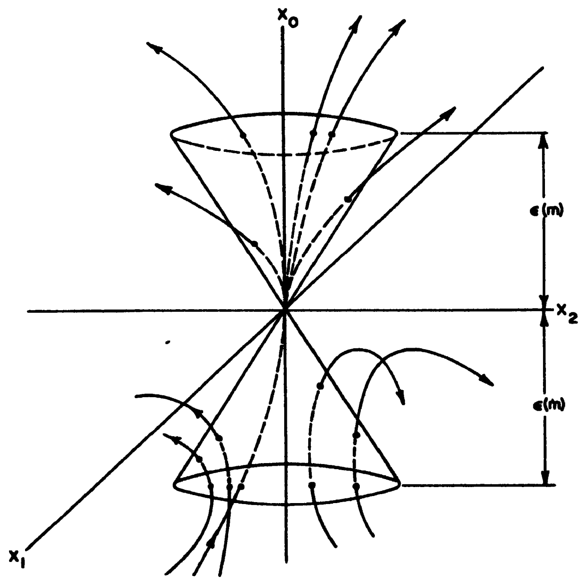


FIGURE 11

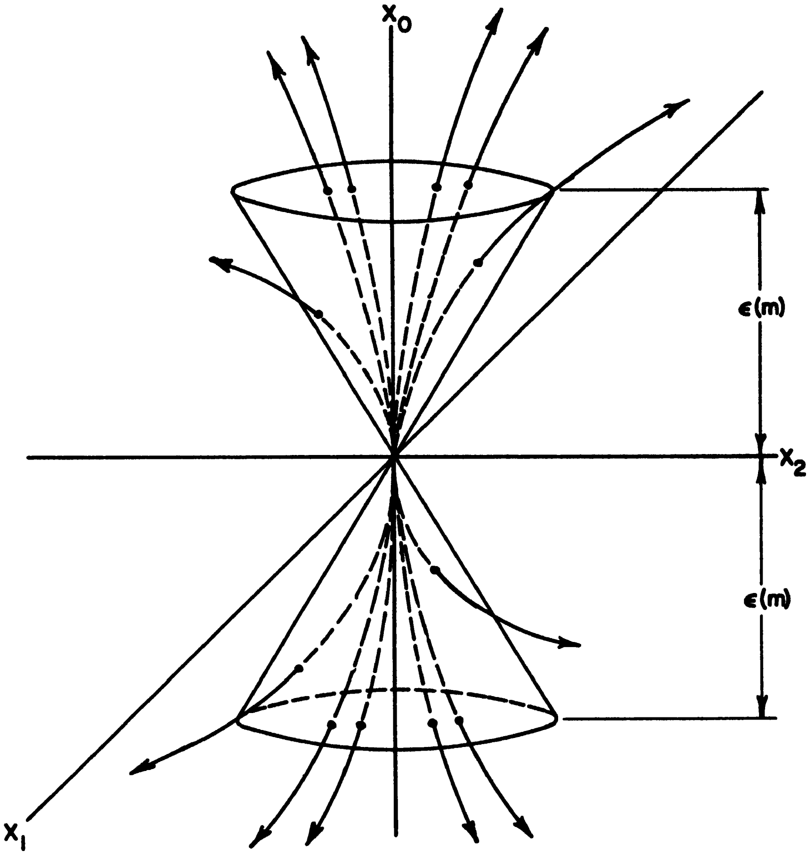


FIGURE 12

6.2. Theorems 1 and 2 complete the description of the local phase portrait of system (12) within the cones $S^{\pm}(m, \epsilon(m))$. We now proceed to investigate the behavior of the trajectories outside these cones.

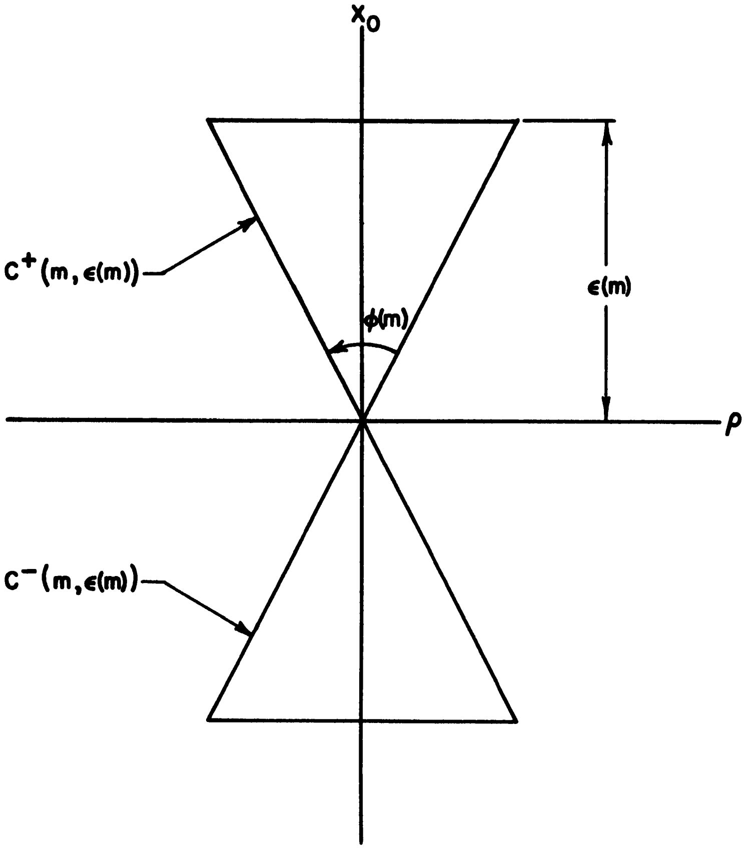


FIGURE 13

Let $m \geq 1$. If $\varphi(m)$ is the "opening" (Figure 13, above) of the come $C^{\pm}(m, \epsilon(m))$, then clearly $\varphi(m) \longrightarrow \pi$ as $m \longrightarrow +\infty$. Furthermore, if $m_1 > m_2$, it follows from the above discussion that $\epsilon(m_1) \leq \epsilon(m_2)$. Thus $\epsilon(m)$ is a decreasing function.

Two distinct cases may arise:

(i) $\exists\, \rho^* > 0$ such that $\rho^* < m\,\epsilon(m)$ for all m.

(ii) $m\,\epsilon(m) \longrightarrow 0$ as $m \longrightarrow +\infty$.

6.3. Suppose case (i) applies. Let B^* be the cylinder defined by

$$B^* = \left\{ x \;\middle|\; \sum_{i=1}^{n} x_i^2 \leq \rho^{*2}; \quad -\rho^*\epsilon(1) \leq x_0 \leq \rho^*\epsilon(1) \right\}.$$

Then, if $Q(x_0, \ldots, x_n)$, $x_0 \neq 0$, is any point contained in B^*, there exists an $m_Q \geq 1$ such that either $Q \in S^+(m_Q, \epsilon(m_Q))$ or $Q \in S^-(m_Q, \epsilon(m_Q))$ (Figure 14). Thus, the phase portrait above and below the $x_0 = 0$ hyperplane is completely determined. It follows from continuity that in this case the $x_0 = 0$ hyperplane is a solution surface. The phase portrait is therefore the same as that of a system (17).

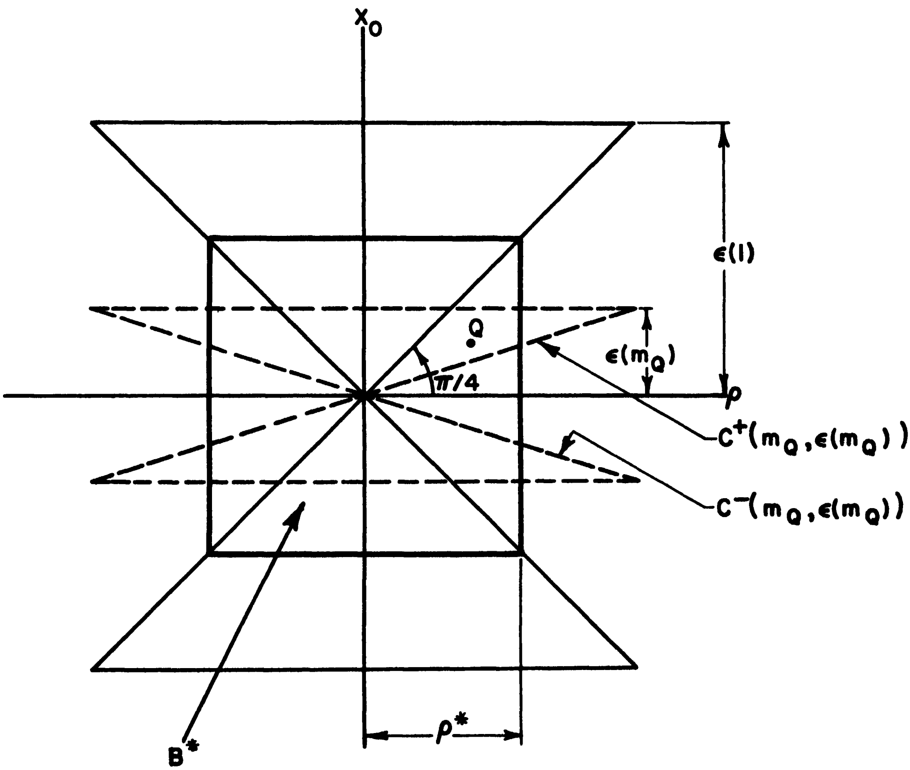


FIGURE 14

Conversely, let the $x_o = 0$ hyperplane be a solution surface. Then it can be shown readily that there exists a cylinder B^* with radius ρ^* and height h^* such that $Q(x_o, \ldots, x_n) \in B^*$, $x_o \neq 0 \Longrightarrow \exists\, m_Q \geq 1$ such that either $Q \in S^+(m_Q, \epsilon(m_Q))$ or $Q \in S^-(m_Q, \epsilon(m_Q))$. It follows therefore that $\rho^* < m\,\epsilon(m)$ for all m. We have thus proved:

THEOREM 3. The $x_o = 0$ hyperplane is a solution surface if, and only if, there exists a $\rho^* > 0$ such that $\rho^* < m\,\epsilon(m)$ for all m. The phase portrait is, in this case, completely determined, and is, in fact, given by one of Figures 3, 5, 6 and 8.

6.4. In general, however, the $x_o = 0$ plane is not an integral surface, and therefore $m\,\epsilon(m) \longrightarrow 0$ as $m \longrightarrow +\infty$. Let Φ be such a system and assume first that k is either even or odd and all the characteristic roots of A have negative real parts.

Let $B(m)$ be the cylinder defined by $\{x \mid \rho \leq m\,\epsilon(m), -\epsilon(m) < x_o < \epsilon(m)\}$ where

$$\rho^2 = \sum_{i=1}^{n} x_i^2$$

and $\epsilon(m)$ is as defined in Theorem 1. Let $R(m)$ be the lateral surface of $B(m)$, i.e., $R(m) = \{x \mid \rho = m\,\epsilon(m), -\epsilon(m) < x_o < \epsilon(m)\}$. Finally, let $\omega(m) = \{x \in B(m) \mid \underline{\rho < m\,\epsilon(m)}, \underline{\rho > mx_o}\}$ or equivalently:

$\omega(m) = \text{Int.}\ \{\overline{B(m)} - [S^+(m, \epsilon(m)) \cup S^-(m, \epsilon(m))]\}$ (Figure 15).

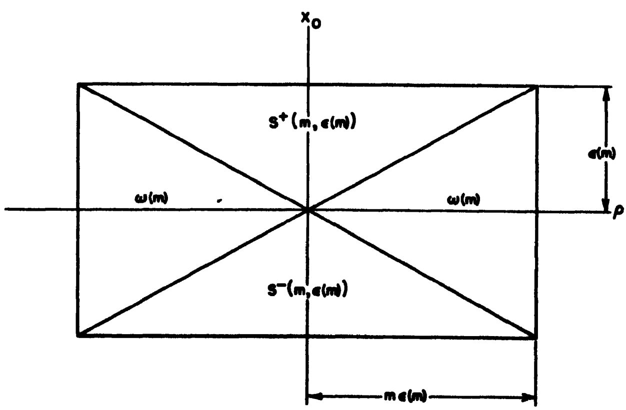


FIGURE 15

Clearly, $m_1 > m_2 \Longrightarrow B(m_1) \subset B(m_2)$ and $\omega(m_1) \subset \omega(m_2)$. Furthermore, by using arguments similar to ones employed in Section 5 (together with the fact that $x_0 < \frac{\rho}{m}$ throughout $\omega(m)$ it can be shown that a large enough $m_0 > 0$ may be picked so that $\dot{\rho} < -\frac{\lambda}{2}\rho$ throughout $\omega(m_0)$. We shall assume henceforth that $m \geq m_0$ and therefore $\dot{\rho} < -\frac{\lambda}{2}\rho$ throughout $\omega(m) \subset \omega(m_0)$.

For the case under consideration we have shown (Theorem 1) that every $x \in C^+(m, \epsilon(m))$ is a point of strict access into the interior of $S^+(m, \epsilon(m))$, and every $y \in C^-(m, \epsilon(m))$ is a point of strict access into the interior of $S^-(m, \epsilon(m))$. Let now $P \in C^+(m, \epsilon(m))$. Then P is a point of strict egress from $\omega(m)$. Since $\dot{\rho} < -\frac{\lambda}{2}\rho$ throughout $\omega(m)$, it follows that if we continue Γ_P from P in the negative direction, Γ_P must intersect $R(m)$ at some point $\tilde{\tilde{P}}$. Let the mapping $P \longrightarrow \tilde{\tilde{P}}$ of $C^+(m, \epsilon(m))$ into $R(m)$ be denoted by f. Then f is a homeomorphism. Since $C^+(m, \epsilon(m))$ is an open n-dimensional manifold, it follows from Brouwer's theorem on the invariance of domain that $O^+_{(m)} = f(C^+(m, \epsilon(m)))$ is open in the n-dimensional manifold $R(m)$.

Similarly, if $Q \in C^-(m, \epsilon(m))$ then Q is a point of strict egress from $\omega(m)$ and if we follow Γ_Q from Q in the negative direction, Γ_Q must intersect $R(m)$ at some point $\tilde{\tilde{Q}}$. The mapping $g : Q \longrightarrow \tilde{\tilde{Q}}$ is a homeomorphism as above, and $O^-(m) = g(C^-(m, \epsilon(m)))$ is open in $R(m)$.

Evidently $O^+(m) \cap O^-(m) = \emptyset$. For if $x \in O^+(m)$ then Γ_x leaves $\omega(m)$ through $C^+(m, \epsilon(m))$, whereas $x \in O^-(m)$ implies that Γ_x leaves $\omega(m)$ through $C^-(m, \epsilon(m))$. Since $C^+(m, \epsilon(m)) \cap C^-(m, \epsilon(m)) = \emptyset$ it is clear that $O^+(m)$ and $O^-(m)$ must be disjoint.

Let $F(m) = R(m) - \{O^+(m) \cup O^-(m)\}$. Then $F(m)$ disconnects $R(m)$ into two parts. Furthermore $F(m)$ is closed in $R(m)$ and clearly also in E_{n+1}.

Let $x \in F(m)$, then Γ_x^+ cannot leave $\omega(m)$ either through $C^+(m, \epsilon(m))$ or through $C^-(m, \epsilon(m))$. Hence $\Gamma_x^+ \subset \omega(m)$ and since $\dot{\rho} < -\frac{\lambda}{2}\rho$ throughout $\omega(m)$, Γ_x must tend to the origin as $t \longrightarrow +\infty$.

Conversely, if $x \in R(m)$ is such that Γ_x tends to the origin as $t \longrightarrow +\infty$ then $x \notin O^+(m) \cup O^-(m)$ and hence $x \in F(m)$.

Suppose $x \in F(m^*)$ for some m^*. Then Γ_x^+ intersects $F(m)$ for all $m \geq m^*$. For suppose $\exists\, m > m^*$ such that $\Gamma_x^+ \cap F(m) = \emptyset$. Then Γ_x^+ can stay in $\omega(m)$ only for a finite time and therefore Γ_x^+ must tend to the origin outside $\omega(m)$. But then Γ_x^+ must finally enter either $S^+(m, \epsilon(m))$ or $S^-(m, \epsilon(m))$. In that case, however, it follows from Theorem 2 that Γ_x^+ has a definite tangent at the origin, and, in fact, this tangent is the x_0-axis. This contradicts the fact that $\Gamma_x^+ \subset \omega(m^*)$.

Hence Γ_x^+ must intersect $F(m)$ for all $m \geq m$ (Figure 16). This, in turn, implies that $\Gamma_x^+ \cap \overline{\{S^+(m, \epsilon(m)) \cup S^-(m, \epsilon(m))\}} = \emptyset$ for all $m \geq m^*$. Since, moreover, $\Gamma_x(t)$ tends to the origin as $t \longrightarrow +\infty$, we conclude that $\Gamma_x(t)$ is tangent to the $x_0 = 0$ hyperplane at the origin.

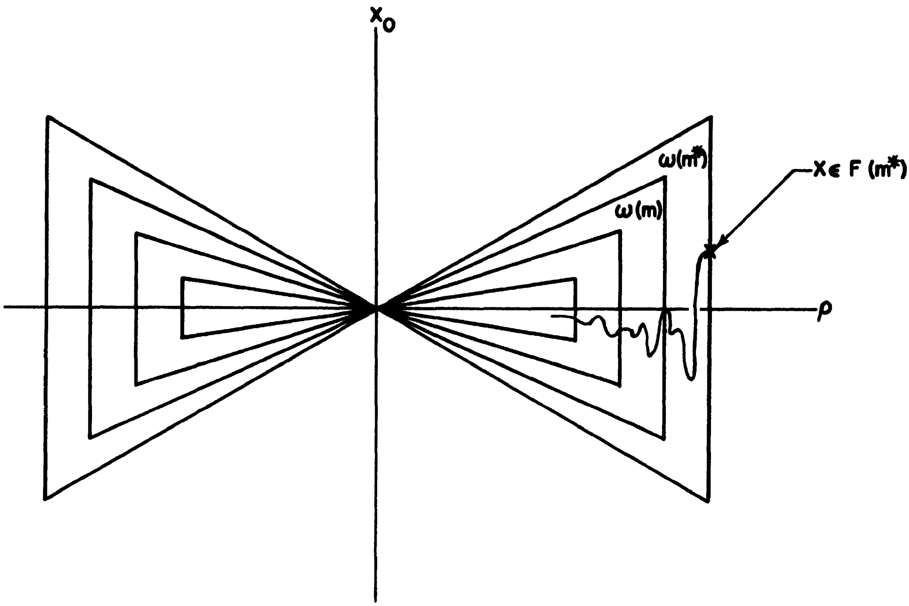


FIGURE 16

A similar situation obtains when all the characteristic roots of A have positive real parts.

We have thus completed the proof of the following:

THEOREM 4. Let all the roots of A have negative (positive) real parts, and let $R(m) = \{x \mid \rho = m\,\epsilon(m),\ -\epsilon(m) < x_0 < \epsilon(m)\}$ where $\rho^2 = \Sigma_{i=1}^n x_i^2$ and $\epsilon(m)$ is as defined in Theorem 1. Then for all m large enough there exists a set $F(m) \subset R(m)$ such that:

(i) $F(m)$ is closed in E_{n+1}

(ii) $F(m)$ disconnects $R(m)$ into two disjoint parts $O^+(m)$, $O^-(m)$.

$(iii)_{\overset{+}{}}$ $\dot{x} \in \overset{+}{O^-}(m) \Longrightarrow \Gamma_x^+ \{\Gamma_x^-\}$ enters $\overset{+}{S^-}(m, \epsilon(m))$ through $\overset{+}{C^-}(m, \epsilon(m))$.

(iv) $x \in F(m) \Longleftrightarrow x \in R(m)$ and $\Gamma_x(t) \longrightarrow 0$ as $t \longrightarrow +\infty$ $\{t \longrightarrow -\infty\}$. Moreover, $x \in F(m) \Longrightarrow \Gamma_x(t)$ is tangent to the $x_0 = 0$ hyperplane at the origin.

ADDED IN PRINT. Condition (iv) of Section 3.8 can be relaxed to read $\dot{x}_0 = x_0^k + {}^*[x_0, \ldots, x_n]_k^0$. All proofs remain unchanged.

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X. ON NON-LINEAR REPULSIVE FORCES[1*]

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1. Introduction and Summary. In this note we shall establish that a mechanical system of n degrees of freedom which is subject to arbitrary (non-linear, non-conservative, non-autonomous, velocity dependent) forces of non-attractive type has a "2n-parameter family" of solutions, of which an "n-parameter subfamily" of solutions is "asymptotic" (in particular, exists and is bounded for $0 \leq t < \infty$), and of which another "n-parameter subfamily" of solutions is unbounded and of monotone increasing magnitude. The sole restriction is that the forces be everywhere continuous, and, if velocity dependent, satisfy a certain growth restriction which is linear in the magnitude of the velocities.

In the autonomous case, there is a simple geometrical interpretation. Let E^k denote real Euclidean k-space. Then in its phase space E^{2n} such a system consists of (i) a single rest point, (ii) positively asymptotic orbits, and (iii) positively divergent orbits. The rest point is a 2n-dimensional saddle point, in the sense that there is an n-dimensional "stable" manifold, passing through the rest point, and such that all orbits originating on it are positively asymptotic to the origin, while for no other orbit does its positive limit set exist.

Let $I_+ : 0 \leq t < \infty$ denote the real half-line and let $f : E^n \times E^n \times I_+ \longrightarrow E^n$ be a map $f = f(x, y, t)$. Consider the vector system of ordinary differential equations

$$\text{(S)} \qquad x'' = f(x, x', t), \qquad (' = d/dt)$$

where the scalar product

$$\text{(R)} \qquad x \cdot f(x, y, t) > 0, \quad \text{unless} \quad x = 0 \ .$$

The condition (R) implies that $f(0, y, t) \equiv 0$ and that $x = 0$ is the only equilibrium solution $(x' = x'' = 0)$ of the system (S) - (R).

Numbered footnotes appear at end of paper.

In the autonomous case $f = f(x, x')$, $f \in C^2$, suppose also that the Jacobian matrix $f_x(0, 0)$ is symmetric. Then in E^{2n} replace (S) by $(S_a) : x' = y, y' = f(x, y)$, and at the origin $(x, y) = (0, 0)$ the system (S_a) has, by (R), exactly n negative characteristic roots and exactly n positive characteristic roots.[2] (In fact, (R) implies that $f_y(0, 0) = 0$ and that $f_x^0(0, 0)$ is positive definite.[3]) Hence well-known theorems (cf. e.g., the recent books of Coddington-Levinson or Lefschetz) imply that the geometrical description given above holds, at least in some sufficiently small neighborhood of the origin. The extension of this description to all of E^{2n} (and in particular, the assertion that the "stable" manifold intersects every hyperplane $x = \text{const.}$) can be effected if and only if the growth of $f(x, y)$ as a function of y is further restricted.

For a simple example, consider a particle sliding under the influence of gravity on a vertical circle (and free to leave the circle at its horizontal extremities). Measuring arc length x from the unstable rest point, and letting $g = 1$, we have

$$\text{(E)} \qquad x'' = f(x), \qquad xf(x) > 0 \quad \text{for} \quad x \neq 0 ,$$

where here $f(x) = \sin x$ for $|x| \leq \pi/2$ and $f(x) = \operatorname{sgn} x$ for $|x| > \pi/2$. By explicit solution of (E), it can be seen that to each initial position x_0 there corresponds

(I) an initial velocity $x'_-(x_0)$ for which the corresponding solution $x(t)$ exists on I_+ and satisfies $|x(t)| \leq |x_0|$; and

(II) an initial velocity $x'_+(x_0)$ for which the corresponding solution $x(t)$ satisfies $|x(t)| \longrightarrow +\infty$ as $t \longrightarrow t_\infty$ (for some $t_\infty \leq \infty$).

For the example (E), the planar phase-portrait can be readily sketched. The origin is a saddle point. In case (I), $(x(t), x'(t)) \longrightarrow (0, 0)$ as $t \longrightarrow \infty$. The assertions (I), (II) are simply that at least one "exceptional integral curve" (i.e., "one through the origin") and also at least one "divergent" integral curve cuts each line $x = x_0$.

For general scalar systems of the type of (E), and more generally, $x'' = f(x, t)$, the assertions (I) - (II) are implied by a theorem of A. Kneser [8]. Hartman and Wintner [4] extended Kneser's theorem to include velocity dependent forces, i.e., they dealt with (S) in the case $n = 1$. (Subsequently [5] they generalized this result to first order systems in such a way as to include [8] and [4].) However, Hartman and Wintner

require that f satisfy not only (R) but also for all y and each $R > 0$

$$\text{(G)}\qquad \begin{aligned} &|f(x, y, t)| \le \varphi_R(|y|), \\ &0 \le t \le R,\ 0 \le |x| \le R \quad , \end{aligned}$$

where $\varphi = \varphi_R$ is such that

$$\text{(Q)}\qquad \int^{\infty} \frac{r}{\varphi(r)}\, dr = +\infty \quad .$$

The ("quadratic") Nagumo condition (G) - (Q) permits e.g., $x'' = (1 + (x')^2)\sin x$, but not $x'' = (1 + (x')^{2+\epsilon})\sin x$ for any $\epsilon > 0$. An example of Nagumo shows that (G) - (Q) is the best possible general restriction (for $n = 1$).

In the scalar autonomous case, there is a simple geometrical explanation of the meaning of condition (G) - (Q). In fact, (S) can be replaced by the first order equation $dy/dx = F(x, y)$, where $F(x, y) \equiv f(x, y)/y$. Then (Q) is just Wintner's general criterion

$$\left(|F| \le \psi,\quad \int^{\infty} \Psi(r)^{-1}\, dr = +\infty\right)$$

for the unrestricted existence of solutions $y = y(x)$; cf. [14] or Ex. 5, p. 61 of Coddington-Levinson's Ordinary Differential Equations (1955). In other words, the exceptional integral curve $y = y(x)$ has less than exponential growth; hence it cuts every vertical line $x = \text{const}$.

Because of the distinctions between ordinary and scalar products, the arguments used for $n = 1$ cannot be generalized to the case $n > 1$. However, for linear systems Wintner [15] generalized Kneser's theorem as follows. Let $x'' = F(t)x$, where the matrix $F(t)$ is continuous on I_+ and non-negative definite; then (I) - (II) holds and there are n linearly independent solutions of each type. Moreover, the same is true ([3]; cf. also [6]) for systems $x'' = G(t)x' + F(t)x$, where F, G are continuous matrices and the symmetric part F^0 of F satisfies $F^0 \ge GG^* \ (\ge 0)$.

The object of this note is to generalize Kneser's theorem to the system (S) - (R), for all $n > 1$, in such a way as to include the linear result of Harman and Wintner ([3]) just quoted.

However, we shall replace (G) - (Q) by the less general condition

$$\text{(L)}\qquad \|f(x, y, t)\| \le C_R + K_R\|y\|$$

for all vectors y and all $R > 0$, and for all $0 \leq t \leq R$ and $\|x\| \leq R$, where the "constants" C_R and K_R depend only on R.

<u>For the system</u> (S) - (R) - (L) <u>the conclusions</u> (I) <u>and</u> (II) <u>hold</u>.

The proof is similar to that of Kneser except that a certain boundary value problem arising in a lemma is more difficult. This problem is reduced to a non-linear integral equation, for whose solutions (L) provides an <u>a priori</u> bound. Then in the manner of Birkhoff-Kellogg, Schauder, and Leray, the Lefschetz fixed-point theorem[4] for compact HLC spaces can be used to solve this integral equation.

Despite considerable effort, I have been unable to relax (L) to (G) - (Q) in the case $n > 1$, or to provide a counter-example; this interesting problem remains open.

The remainder of this note is similar to Part Two of my 1955 dissertation, which was written at The Johns Hopkins University under the direction of Professor Wintner. I wish to thank Professor Wintner for suggesting this topic, and to thank both him and Professor Hartman for their assistance and advice. I also wish to thank Professors Lefschetz, Markus, and Barocio for the illuminating geometrical discussion of this result which was reproduced above; this discussion was developed following my presentation of the result to Professor Lefschetz's Princeton seminar.

2. To give a more precise formulation of the results to be obtained, let x be the coordinate vector $(x_1, \ldots, x_n)$ and let $f(x, x', t)$ be a real vector function $(f_1, \ldots, f_n)$ which is continuous on the half-space

$$(1) \qquad 0 \leqq t < \infty, \quad -\infty < x_i < +\infty, \quad -\infty < x_i' < +\infty, \quad (i = 1, \ldots, n) \quad .$$

As a generalization of the non-negative definiteness of the coefficient matrix F^0 in the simple linear case $G \equiv 0$, suppose that

$$(2) \qquad x \cdot f(x, x', t) \geqq 0$$

for all x, x', t on (1). We may then say that the differential equations

$$(3) \qquad x'' = f(x, x', t)$$

represent a mechanical system in which, by virtue of (2), the forces have no "centrally attractive" component.

Now put $r = \|x\|^2 \equiv x \cdot x$. In addition to (2), we shall assume that for every $r_0 > 0$ and $T > 0$ there exist positive numbers $C = C(r_0, T)$ and $K = K(r_0, T)$ such that

(4) $$\|f(x, x', t)\| < C + K\|x'\|$$

for t, x satisfying

(5a) $0 \leqq t \leqq T$, (5b) $0 \leqq r \leqq r_o$,

and for all x'. Since $C(r_o, T)$ and $K(r_o, T)$ can always be taken to be monotone increasing and continuous in T, we shall assume for later convenience that this has been done.

Next, note that $r'(t) = 2x\cdot x'$, $r''(t) = 2x\cdot x'' + 2x'\cdot x'$, and by (2) and (3) we have

(6) $$r'' \geqq 0, \quad \text{where} \quad r = \|x\|^2 \quad .$$

This fundamental inequality implies that <u>the squared length</u> $r(t)$ <u>of any solution vector of</u> (3) - (2) <u>is convex toward the t-axis</u>. The results which follow actually depend on (6) rather than (2). In the linear case, $F^o \geqq GG^*$ is sufficient for (6); cf. [3]. Also it is clear that in the linear case conditions (4) - (5) are satisfied.

By means of (4) and (6) we shall show (cf. Section 3) that all local solutions of (3) at $t = 0$ can be continued onto a ("maximal") interval

(7a) $$0 \leqq t < t^o \quad ,$$

but not onto $0 \leqq t \leqq t^o$, where $t^o < \infty$ if and only if

(7b) $$r(t) \longrightarrow \infty \quad \text{as} \quad t \longrightarrow t^o \quad .$$

Consequently, all solutions of (3) are unrestricted except those for which (7b) holds for some $t^o < \infty$; in particular, solutions for which $r(t)$ is monotone non-increasing exist on (7a) with $t^o = \infty$.

Let us define a <u>divergent solution</u> of (3) as one which exists on (7a) and satisfies (7b) for some $t^o \leqq \infty$. On the other hand, we shall say that a solution is <u>asymptotic</u> if $r(t)$ satisfies

(8) $$r(t) > 0, \quad r'(t) \leqq 0, \quad r''(t) \geqq 0 \quad \text{for} \quad 0 \leqq t < \infty \quad ,$$

so that, in particular,

$$\lim_{t \to \infty} r(t) \text{ exists } (< \infty) \text{ and } \lim_{t \to \infty} r'(t) = 0 \quad .$$

Finally, let $x(t;\ x_0,\ x'_0)$ denote any continuation onto its maximal interval (7a) of any solution of (3) which is defined at $t = 0$ and satisfies there

$$(9) \qquad x(0;\ x_0,\ x'_0) = x_0, \quad x'(0;\ x_0,\ x'_0) = x'_0 \quad .$$

(This solution is not in general uniquely determined by the initial conditions (9).)

Using the above terminology, we now state our result as follows.

THEOREM. Let f be continuous on (1) and satisfy (4) - (5). Then corresponding to every initial position x_0, there exist two vectors $x'_+(x_0)$ and $x'_-(x_0)$ and corresponding solutions of (3) which are such that

(10) every $x(t;\ x_0,\ x'_+(x_0))$ is divergent $(0 \leqq t < t^0 \leqq \infty)$,

(11) some $x(t;\ x_0,\ x'_-(x_0))$ is asymptotic $(0 \leqq t < \infty)$.

In the linear case (10) - (11) imply that there are n linearly independent divergent solutions and n linearly independent asymptotic solutions.

REMARK. It is clear from the alternative described in (7a), (7b) that an $x'_+(x_0)$ can be selected readily. (In fact, we need only let x'_+ be any vector such that $x'_+ \cdot x_0 > 0$; then $r'(0) > 0$, and (by the convexity of $r(t)$) (7b) must hold for some $t^0 \leqq \infty$.) We now establish this alternative.

3. According to a result of Wintner [14], all continuations of a local solution $x(t;\ x_0,\ x'_0)$ are unrestricted unless there exists a $t^0 < \infty$ such that

$$\lim\left\{r(t) + \|x'(t)\|^2\right\} = \infty, \quad \text{as} \quad t \longrightarrow t^0 \quad .$$

Thus, if $r(t)$ is bounded on $0 \leqq t < t^0$ (so that (5b) holds there for some r_0), then $\lim\|x'(t)\| = \infty$ as $t \longrightarrow t^0$. From this we derive a contradiction with (4), by means of the following lemma.

LEMMA 1. Let $x(t)$ be a solution of (3), defined on (5a), where $f(x,\ x',\ t)$ is continuous on (1) and

has the property (4) - (5) (but not necessarily (2) - (6)). Then if $x(t)$ satisfies (5b), there exists a constant $D = D(r_o, T)$ such that $x'(t)$ satisfies

$$0 \leqq \|x'(t)\| \leqq D(r_o, T) \quad . \tag{12}$$

This a priori bound for $x'(t)$ follows from modification of a device of Hartman and Wintner [3]. Since $x'(t)$ is continuous on (5a) there exists some t_* in (5a) such that $\|x'(t_*)\| = \max\|x'(t)\|$ on (5a). Now let

$$|\tau| = 1/(mK(r_o, T)) \quad , \tag{13}$$

where $m = m(r_o, T) \geqq n$ is so large that $|\tau| \leqq T/2$, and choose $\operatorname{sgn} \tau$ such that $t = t_* + \tau$ lies in (5a). Then by the mean value theorem there exist numbers θ_i $(i = 1, \ldots, n)$ lying in (5a) and such that if we put $x''(\theta) \equiv (x_1''(\theta_1), \ldots, x_n''(\theta_n))$, we have

$$x(t_* + \tau) = x(t_*) + \tau x'(t_*) + \frac{1}{2}\tau^2 x''(\theta) \quad . \tag{14}$$

But by (3) and (4), $\|x''(\theta)\| \leqq nC + nK\|x'(t_*)\| \leqq nC + mK\|x'(t_*)\|$, so that by (5) and (14),

$$|\tau|\,\|x'(t_*)\| \leqq 2(r_o)^{1/2} + \frac{1}{2}mK\tau^2\|x'(t_*)\| + \frac{1}{2}nC\tau^2$$

i.e., $|\tau|(1 - \frac{1}{2}mK|\tau|)\|x'(t_*)\| \leqq 2(r_o)^{1/2} + \frac{1}{2}nC\tau^2$, which from (13) gives (12) with $D(r_o, T) \equiv 4m(r_o)^{1/2}K(r_o, T) + nC(r_o, T)/mK(r_o, T)$. This proves the lemma and so the alternative (7a) - (7b).

4. It remains only to select $x_-'(x_o)$ and a corresponding solution (which is not unique) in such a manner as to satisfy (11). In our construction of an asymptotic solution of (3), the following property (J) proves to be fundamental:

(J). To every vector x_o and every positive number T there corresponds a solution $x_T(t)$ of (3), defined on (5a), and satisfying

$$x_T(0) = x_o, \qquad x_T(T) = 0 \quad . \tag{15}$$

Hereafter we shall refer to (J) as the joinability property for (5a). Because the successive approximations by means of which Picard [9] established local joinability (under certain smoothness assumptions for a

system similar to (3) do not converge for large T, it will be necessary to prove in some other manner that (6) implies joinability in the large. To this end we shall employ the following lemma (which is independent of (2), (4)).

LEMMA 2. Suppose that $f(x, x', t)$ is continuous and bounded, say $|f| \leqq M$, on

$$(16) \qquad 0 \leqq t \leqq T, \quad -\infty < x_i < +\infty, \quad -\infty < x_i' < +\infty, \qquad (i = 1, \ldots, n) \quad .$$

Then (3) has the joinability property on (5a).

As is well known ([10]), the lemma is true if and only if the integral equation

$$(17) \qquad x(t) = x_0(1 - t/T) - \int_0^T \varphi(u, t)f(x(u), x'(u), u)du$$

(where $\varphi = u(T - t)/T$ on $0 \leqq u \leqq t \leqq T$, $\varphi = t(T - u)/T$ on $0 \leqq t < u \leqq T$, and $0 \leqq \varphi(u, t) \leqq T/4$) has a solution $x = x_T(t)$ on (5a).

For $n = 1$, Lemma 2 was stated by Birkhoff and Kellogg [1] as a corollary of their fixed-point theorem (cf. the elementary proof of Lemma 2 by Scorza-Dragoni [12]). It seems possible to extend their methods to include $n \geqq 2$, but such an extension is not immediate. A (slightly weaker) form of their fixed-point theorem was later discovered independently by Caccioppoli [2], and his proof for scalars clearly can be extended to apply to vector functions as well. On this basis, Scorza-Dragoni [13] stated the lemma for $n = 2$. As a matter of fact, the lemma for $n \geqq 2$ is an obvious immediate consequence of Schauder's fixed-point theorem [11]. For (i) the integral transformation on the right-hand side of (17) is a continuous mapping $\mathcal{F}$ of the Banach space $\underline{B}$ of class C^1 vector functions on (5a), into itself, where (by definition), (ii) $\underline{B}$ is complete in the topology of uniform convergence of the functions and their first derivatives, and (iii) $\mathcal{F}$ is "completely continuous", in the sense that the closure of $\mathcal{F}(\underline{B})$ is compact. ($\mathcal{F}: K \longrightarrow K = \widehat{\mathcal{F}(\underline{B})}$, an absolute retract.[5])

Now let x_0 and $T > 0$ be arbitrary and consider any vector function $f^*(x, x', t)$ which is identical with the $f(x, x', t)$ of (1), (3), and (4) - (5) on the product cylinder

$$(18) \qquad 0 \leqq t \leqq T, \quad 0 \leqq r \leqq r_0, \quad 0 \leqq \|x'\| \leqq D(r_0, T) \quad ,$$

and which on the half-space (1) is continuous and has the same bound, M, as does $f(x, x', t)$ on (18). By Lemma 2 the system $x'' = f^*(x, x', t)$ has the joinability property on (5a). But by (6), any solution $x_T(t)$ satisfying (15) also satisfies (8), and so by Lemma 1 lies entirely within the cylinder (18). This shows (cf. the similar argument in Kamke [7], p. 129) that the introduction of f^* was unnecessary, i.e., that $f(x, x', t)$ satisfies joinability on (5a). But because $T > 0$ was arbitrary, this proves the unrestrictedness of the joinability property of (3).

5. The remainder of the proof now proceeds as in [15]. Let x_0 be fixed arbitrarily and consider the sequence of solutions $x_T(t)$ of (3) satisfying (15) for $T = 1, 2, 3, \ldots$. From (15), we have

$$(19) \qquad r_m(0) = r_0 = x_0 \cdot x_0, \quad r_m(m) = 0, \quad (m = 1, 2, \ldots) ,$$

where

$$(20) \qquad r_m = r_m(t) = \|x_m(t)\|^2 .$$

But (6) and (19) imply that $r_m(t)$ is monotone for $0 \leqq t \leqq m$. Consequently we have also

$$(21) \qquad r_m(t) > 0, \quad r_m'(t) \leqq 0, \quad r_m''(t) \geqq 0, \quad \text{if} \quad 0 \leqq t < m .$$

Now the first of the conditions (19) and the second of the inequalities (21) imply that the functions $r_m(t)$ are uniformly bounded by r_0 on every fixed, bounded t-interval. In view of (20) the same holds for the vectors $x_m(t)$, and by Lemma 1 and (3), for the vectors $x_m'(t)$, $x_m''(t)$ as well (the function $f(x, x', t)$ being continuous on the compact cylinder (18) for fixed, bounded T).

Consequently, the sequences $\{x_m(t)\}$, $\{x_m'(t)\}$ are uniformly bounded and are equicontinuous on every fixed, bounded t-interval. By the Ascoli-Arzela selection principle, we can choose an increasing sequence of integers $m = m(k)$ $(k = 1, 2, \ldots)$, such that as $k \longrightarrow \infty$, $x_{m(k)}(t)$ and $x_{m(k)}'(t)$ converge (uniformly on every fixed, bounded interval of the half-line $0 \leqq t < \infty$) to limit functions $x(t)$ and $x'(t)$; furthermore, $x'(t)$ is the derivative of $x(t)$.

Since every $x_{m(k)}(t)$ is a solution of (3), it is clear that $x(t)$ is an unrestricted solution of (3), and that, by (20) and (21), $r(t)$ satisfies (8). The desired vector $x_-'(x_0)$ is then given by $x'(0)$, and the proof of (11) is complete.

In the linear case, let ${}^m x_0$, $(m = 1, \ldots, n)$, be n linearly

independent initial position vectors and let ${}^m x'_-$ be corresponding initial velocities which determine (unique) asymptotic solutions; for simplicity, fix ${}^m x_0$, $(m = 1, \ldots, n)$, as the n unit vectors $(1, 0, \ldots, 0)$, $(0, 1, 0, \ldots, 0)$, $\ldots$. Next let ${}^m x'_+ = {}^m x_0$ (so that ${}^m r'_+(0) = {}^m x'_+ \cdot {}^m x_0 > 0$). Now since ${}^m r'_-(0) = {}^m x'_- \cdot {}^m x_0 < 0$, the n asymptotic solutions are linearly independent of the n divergent solutions.

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FOOTNOTES

1 Results obtained while the author held a National Science Foundation Fellowship at The Johns Hopkins University. Extended and revised at Princeton University (under Naval Research Contract N6ori-105, Task Order v, Project NR043-942) and at RIAS, Inc. (Baltimore).

2 Similarly for real parts of roots when $f_x(0, 0) \neq f_x^0(0, 0)$.

3 By A^0 denote symmetric part of a matrix A.

4 If K is an AR (absolute retract) compactum and $\mathcal{F} : K \longrightarrow K$ is any self-map of K, then $\mathcal{F}$ has at least one fixed point $\xi = \mathcal{F}(\xi)$ in K. Cf. e.g., S. Lefschetz, Topics in Topology, Princeton University Press (1942), p. 116 and p. 119.

5 Here $\hat{D}$ denotes the convex closure of a set $D \subset \underline{\underline{B}}$. If the closure of D is compact, then $\hat{D}$ is an AR.

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